REPORTS ON MATHEMATICAL LOGIC
44 (2009), 3–18

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COMPLETENESS OF RELEVANT MODAL LOGICS WITH DISJUNCTIVE RULES

A b s t r a c t. Disjunctive rules are known to validate material implication principles, which may not hold in weaker relevant logics. On the other hand, they prevent the oddity of not preserving truth at the base world, from which many weaker relevant logics suffer. The same holds for relevant modal logics. This paper proves completeness of relevant modal logics with disjunctive rules.

1. Introduction

A modal logic based on non-classical logic has a long tradition. Examples include intuitionistic modal logics, relevant modal logics, or more generally, substructural modal logics. Indeed, the exponentials of linear logic can be viewed as S4-like modalities.

Received 24 October 2005

Keywords: relevant modal logic, disjunctive rules, Routley-Meyer semantics, completeness

The usual Routley-Meyer semantics for relevant logics employs a frame $\langle O, W, R,^* \rangle$. Here W is a set of all worlds and O, a non-empty subset of W, is a set of all *base* worlds. We have given an intuitive interpretation of W as a class of situations, and that of O as the subclass of regular situations in which all theorems hold. Further, a ternary relation R on W is employed the following condition for implication:

$$a \models A \to B \quad \text{iff} \quad \text{for all } b, c \in W,\ Rabc \ \&\ b \models A \Rightarrow c \models B.$$

Take some weak relevant logics such as **B**. Suppose now that both $A \to B$ and A are true at $a \in W$. Still B may not be true at a, unless some conditions on R, such as $Raaa$ for all $a \in W$, are imposed. Thus, Modus Ponens does not preserve truth at $a \in W$. On the other hand, we may expect that Modus Ponens preserve truth at a base world $a \in O$, but this idea is also defeated in fact.

In Routley-Meyer semantics for certain weak relevant logics an odd situation arises: some rules do not preserve truth at the base world. To avoid that, adding *disjunctive rules* has been considered. The following disjunctive rules were discussed in Section 4.8 of [5].

$$\frac{C \vee (A \to B) \quad C \vee A}{C \vee B} \qquad \frac{E \vee (A \to B) \quad E \vee (C \to D)}{E \vee ((B \to C) \to (A \to D))}$$

$$\frac{C \vee (A \to\sim B)}{C \vee (B \to\sim A)} \qquad \frac{C \vee A}{C \vee \sim (A \to\sim A)} \qquad \frac{C \vee (\sim A \to A)}{C \vee A}$$

Adding suitable disjunctive rules makes it possible to introduce *reduced* Routley-Meyer semantics, in which all rules preserve truth at the base world. Further, as mentioned in [1], the disjunctive meta-rule of the form if '$A \vdash B$' then '$C \vee A \vdash C \vee B$', provides a simpler way of adding disjunctive rules to sentential logic than adding each disjunctive rule individually as in [5]. In fact, many relevant logics which are complete with respect to simplified semantics (a kind of reduced Routley-Meyer semantics, see e.g., [3] and [4]) contain the disjunctive meta-rule. It is also known (see e.g., Section 4.8 of [5]) that there is a frame condition corresponding to each disjunctive rule in unreduced Routley-Meyer semantics.

The truth preserving problem at the base world also arises in relevant modal logics. As can be seen from the soundness argument for basic relevant modal logic $\mathbf{B.C}_{\Box\Diamond}$ introduced in [6], the following rules of inference concerned with modalities do not necessarily preserve truth at the base world:

$$\frac{A \to B}{\Box A \to \Box B} \qquad \frac{A \to B}{\Diamond A \to \Diamond B}.$$

To make the rules of inference preserve truth at the base world, disjunctive rules need to be added for *regular* relevant modal logics that contain neither $(A \to B) \to (\Box A \to \Box B)$ nor $(A \to B) \to (\Diamond A \to \Diamond B)$; and for *normal* relevant modal logics not containing $A \to \Box A$ (for definitions of 'regular' and 'normal' see [6]). This paper deals with completeness of relevant modal logics with the following disjunctive rules:

$$\frac{C \vee (A \to B)}{C \vee (\Box A \to \Box B)} \qquad \frac{C \vee (A \to B)}{C \vee (\Diamond A \to \Diamond B)} \qquad \frac{C \vee A}{C \vee \Box A} \qquad \frac{C \vee \blacklozenge^k \blacksquare^l A}{C \vee \blacksquare^m \blacklozenge^n A},$$

where k, l, m, n are non-negative integers.

2. Disjunctive rules and logics

The language of relevant modal logics consists of (i) propositional variables; (ii) logical connectives $\to, \wedge, \vee$ and $\sim$; and (iii) modal operators $\Box$ and $\Diamond$. Formulas are defined in the usual way, and are denoted by capital letters A, B, C, etc. Further, non-empty sets of formulas are denoted by capital Greek letters Σ, Γ, Δ, etc. When necessary, we use $'$ or subscripts for capital letters and capital Greek letters. Prop and Wff will denote the set of all propositional variables and of formulas, respectively. Moreover, we introduce the following abbreviations:

$$\boxdot A \overset{\text{def}}{=} \sim \Diamond \sim A, \qquad\qquad \dot{\Diamond} A \overset{\text{def}}{=} \sim \Box \sim A.$$

For non-negative integer n, we define the abbreviations $\blacksquare^n$ and $\blacklozenge^n$. Informally, $\blacksquare^n$ denotes a sequence consisting of n of either $\Box$ or $\boxdot$, and $\blacklozenge^n$ denotes a sequence consisting of n of either $\Diamond$ or $\dot{\Diamond}$. Formally, $\blacksquare^n$ and $\blacklozenge^n$ are defined inductively as follows.

(i) $\blacksquare^0 A$ is A.

(ii) For $n > 0$, $\blacksquare^n A$ is either $\Box \blacksquare^{n-1} A$ or $\boxdot \blacksquare^{n-1} A$.

(iii) $\blacklozenge^0 A$ is A.

(iv) For $n > 0$, $\blacklozenge^n A$ is either $\Diamond\blacklozenge^{n-1}A$ or $\Diamonddot\blacklozenge^{n-1}A$.

For instance, $\blacksquare^2$ can be either of $\Box\Box$, $\boxdot\Box$, $\Box\boxdot$ or $\boxdot\boxdot$.

We start with a list of axioms and rules of inference, from which the logics without disjunctive rules will be defined.

(a) Axioms

(A1) $A \to A$

(A2) $A \wedge B \to A$

(A3) $A \wedge B \to B$

(A4) $(A \to B) \wedge (A \to C) \to (A \to B \wedge C)$

(A5) $A \to A \vee B$

(A6) $B \to A \vee B$

(A7) $(A \to C) \wedge (B \to C) \to (A \vee B \to C)$

(A8) $A \wedge (B \vee C) \to (A \wedge B) \vee C$

(A9) $\sim\sim A \to A$

(A10) $(A \to\sim B) \to (B \to\sim A)$

(A11) $(A \to B) \wedge (B \to C) \to (A \to C)$

(A12) $(A \to B) \to ((B \to C) \to (A \to C))$

(A13) $(A \to B) \to ((C \to A) \to (C \to B))$

(A14) $A \to ((A \to B) \to B)$

(A15) $(A \to\sim A) \to\sim A$

(A16) $(A \to (A \to B)) \to (A \to B)$

(A17) $A \vee \sim A$

(A18) $A \wedge (A \to B) \to B$

(b) Rules of inference

$$\text{(R1)}\frac{A \to B \quad A}{B} \qquad \text{(R2)}\frac{A \quad B}{A \wedge B}$$

$$\text{(R3)}\frac{A \to B \qquad C \to D}{(B \to C) \to (A \to D)} \qquad \text{(R4)}\frac{A \to\sim B}{B \to\sim A}$$

$$\text{(R5)}\frac{A}{(A \to B) \to B}$$

The basic relevant logic **B** consists of the axioms (A1)–(A9) and the rules of inference (R1)–(R4). We recall below some relevant logics extending **B** that have become standard. We write $\mathbf{L}+X$ for a logic obtained from **L** by adding an axiom or a rule of inference X. $\mathbf{G} = \mathbf{B} + $ (A17), $\mathbf{DW} = \mathbf{B} +$ (A10), $\mathbf{DJ} = \mathbf{DW} +$ (A11), $\mathbf{DK} = \mathbf{DJ} +$ (A17), $\mathbf{DL} = \mathbf{DK} +$ (A15), $\mathbf{DA} = \mathbf{DL} +$ (A18), $\mathbf{TW} = \mathbf{DW} +$ (A12), $\mathbf{TJ} = \mathbf{TW} +$ (A11), $\mathbf{TK} = \mathbf{TJ} +$ (A17), $\mathbf{TL} = \mathbf{TK} +$(A15), $\mathbf{T} = \mathbf{TL} +$(A16), $\mathbf{C} = \mathbf{TW} +$(A18), $\mathbf{EW} = \mathbf{TW} +$(R5), $\mathbf{E} = \mathbf{T} +$ (R5), $\mathbf{RW} = \mathbf{TW} +$ (A14) and $\mathbf{R} = \mathbf{RW} +$ (A16). From now on, **L1** will stand for any logic defined above. Note that (A13) is derivable in logics containing both (A10) and (A12).

The following rules are called *disjunctive rules.*

$$\text{(DR1)}\frac{C \vee (A \to B) \quad C \vee A}{C \vee B} \qquad \text{(DR2)}\frac{E \vee (A \to B) \quad E \vee (C \to D)}{E \vee ((B \to C) \to (A \to D))}$$

$$\text{(DR3)}\frac{C \vee (A \to\sim B)}{C \vee (B \to\sim A)} \qquad \text{(DR4)}\frac{C \vee A}{C \vee \sim (A \to\sim A)}$$

$$\text{(DR5)}\frac{C \vee (\sim A \to A)}{C \vee A}$$

Now, we let $\mathbf{L1} \oplus Y$ stand for the logic obtained from **L1** by adding a non-empty set Y of disjunctive rules. Any such logic will be denoted by **L2**, except in cases where **L2** is equivalent to **L1**; then we will use **L1** rather than **L2**. In other words, if all disjunctive rules in **L2** are derivable in **L1**, then **L2** is regarded as **L1**. Such situations arise if **L2** contains: (1) (A18) and (DR1); (2) (A12), (A13) and (DR2); (3) (A10) and (DR3); (4) (A15) and (DR4); or (5) (A15) and (DR5). Further, it is known that if **L1** is **B**, **DW** or **DJ**, then $\mathbf{L1} \oplus \{(\text{DR1}), (\text{DR2}), (\text{DR3})\}$ is equivalent to **L1**. (See p.278 of [1].)

Now we present relevant modal logics discussed in this paper. Again, we start with a list of axioms and rules of inference.

(a) All axioms of **L1** (or **L2**).

(A19) $\Box A \wedge \Box B \to \Box(A \wedge B)$

(A20) $\Diamond(A \vee B) \to \Diamond A \vee \Diamond B$

(A21) $\Box(A \to B) \to (\Box A \to \Box B)$

(A22) $\Box(A \to B) \to (\Diamond A \to \Diamond B)$

(GA(k,l,m,n)) For any non-negative integers k,l,m,n:
$\blacklozenge^k \blacksquare^l A \to \blacksquare^m \blacklozenge^n A$

(b) All rules of inference of **L1** (or **L2**).

$$(\mathrm{R6})\frac{A \to B}{\Box A \to \Box B} \qquad (\mathrm{R7})\frac{A \to B}{\Diamond A \to \Diamond B} \qquad (\mathrm{R8})\frac{A}{\Box A}$$

The relevant modal logic $\mathbf{L1.C}_{\Box\Diamond}$ (or $\mathbf{L2.C}_{\Box\Diamond}$) is obtained from **L1** (or **L2**) by adding (A19), (A20), (R6) and (R7); the relevant modal logic $\mathbf{L1.K}_{\Box\Diamond}$ (or $\mathbf{L2.K}_{\Box\Diamond}$) is obtained from $\mathbf{L1.C}_{\Box\Diamond}$ (or $\mathbf{L2.C}_{\Box\Diamond}$) by adding (A21), (A22) and (R8). Note that both (R6) and (R7) are then derivable in $\mathbf{L1.K}_{\Box\Diamond}$ and $\mathbf{L2.K}_{\Box\Diamond}$.

The following rules are also called *disjunctive rules*.

$$(\mathrm{DR6})\frac{C \vee (A \to B)}{C \vee (\Box A \to \Box B)} \qquad (\mathrm{DR7})\frac{C \vee (A \to B)}{C \vee (\Diamond A \to \Diamond B)} \qquad (\mathrm{DR8})\frac{C \vee A}{C \vee \Box A}$$

(GDR(k,l,m,n)) For any non-negative integers k,l,m,n : $\dfrac{C \vee \blacklozenge^k \blacksquare^l A}{C \vee \blacksquare^m \blacklozenge^n A}$

Further, let **L3** denote a logic obtained from either $\mathbf{L1.C}_{\Box\Diamond}$ or $\mathbf{L1.K}_{\Box\Diamond}$ by adding zero or more of (GA(k,l,m,n)), and let **L4** denote a logic obtained:

(i) from $\mathbf{L1.C}_{\Box\Diamond}$ or $\mathbf{L1.K}_{\Box\Diamond}$ by adding zero or more of (GA(k,l,m,n)) and one or more of '(DR6), (DR7), (DR8) and (GDR(k,l,m,n))';

(ii) from $\mathbf{L2.C}_{\Box\Diamond}$ or $\mathbf{L2.K}_{\Box\Diamond}$ by adding zero or more of '(GA(k,l,m,n)), (DR6), (DR7), (DR8) and (GDR(k,l,m,n))'.

If all disjunctive rules can be derived in **L4**, then the **L4** is regarded as **L3**. Roughly speaking, **L4** contains disjunctive rules explicitly but **L3** only implicitly (if at all).

3. Semantics and soundness

We use $\&, \Rightarrow, \forall$ and $\exists$ to denote respectively conjunction, implication, universal quantifier and existential quantifier in the metalanguage. We omit parentheses by assuming that $\forall, \exists$ bind stronger than &, and that & binds stronger than $\Rightarrow$.

A $\mathbf{B.C}_{\Box\Diamond}$-*frame* is a 6-tuple $\langle O, W, R, S_\Box, S_\Diamond, ^* \rangle$ where (a) W is a set of all worlds, (b) O is a non-empty subset of W, (c) R is a ternary relation on W, (d) $S_\Box$ and $S_\Diamond$ are binary relations on W, and (e) * is an unary operation on W. To simplify the notation, we define binary relations $\leq$, $S_{\boxdot}$ and $S_{\dot\Diamond}$ on W as follows. For all $a, b \in W$:

$$a \leq b \overset{\text{def}}{\iff} \exists c (c \in O \;\&\; Rcab), \qquad S_{\boxdot}ab \overset{\text{def}}{\iff} S_\Diamond a^* b^*, \qquad S_{\dot\Diamond}ab \overset{\text{def}}{\iff} S_\Box a^* b^*.$$

A $\mathbf{B.C}_{\Box\Diamond}$-frame $\langle O, W, R, S_\Box, S_\Diamond, ^* \rangle$ satisfies the following postulates. For all $a, b, c, d \in W$:

(p1) $a \leq a$

(p2) $a \leq b \;\&\; Rbcd \Rightarrow Racd$

(p3) $a \leq b \Rightarrow b^* \leq a^*$

(p4) $a^{**} = a$

(p5) $a \leq b \;\&\; S_\Box bc \Rightarrow S_\Box ac$

(p6) $S_\Box ab \;\&\; b \leq c \Rightarrow S_\Box ac$

(p7) $a \leq b \;\&\; S_\Diamond ac \Rightarrow S_\Diamond bc$

(p8) $S_\Diamond ab \;\&\; c \leq b \Rightarrow S_\Diamond ac.$

Any $\mathbf{B.C}_{\Box\Diamond}$-frame can be endowed with a valuation function to define a model. Formally, a $\mathbf{B.C}_{\Box\Diamond}$-*model* on a $\mathbf{B.C}_{\Box\Diamond}$-frame $\mathfrak{F} = \langle O, W, R, S_\Box, S_\Diamond, ^* \rangle$ (later on we will just say a $\mathbf{B.C}_{\Box\Diamond}$-model) is a 7-tuple $\langle O, W, R, S_\Box, S_\Diamond, ^*, v \rangle$ such that $\mathfrak{F}$ is a $\mathbf{B.C}_{\Box\Diamond}$-frame and v is a mapping from $\mathsf{Prop} \times W$ to $\{\mathsf{t}, \mathsf{f}\}$, called a *valuation* on $\mathfrak{F}$, which satisfies the following *hereditary condition*. For all $a, b \in W$ and all $p \in \mathsf{Prop}$:

$$a \leq b \;\&\; v(p, a) = \mathsf{t} \Rightarrow v(p, b) = \mathsf{t}.$$

Given a $\mathbf{B.C}_{\Box\Diamond}$-model $\langle O, W, R, S_\Box, S_\Diamond, ^*, v \rangle$, we define the *interpretation I associated with v*. Informally, interpretation tells us which formulas are true at which points of W. Formally, a mapping I from $\mathsf{Wff} \times W$ to $\{\mathsf{t}, \mathsf{f}\}$ is defined inductively as follows:

i. for any $p \in \mathsf{Prop}$, $I(p, a) = \mathsf{t}$ iff $v(p, a) = \mathsf{t}$

ii. $I(A \wedge B, a) = \mathsf{t}$ iff $I(A, a) = \mathsf{t} \;\&\; I(B, a) = \mathsf{t}$

iii. $I(A \vee B, a) = \mathsf{t}$ iff $I(A, a) = \mathsf{t}$ *or* $I(B, a) = \mathsf{t}$

iv. $I(A \to B, a) = \mathsf{t}$ iff
$\forall b \in W \forall c \in W(Rabc \;\&\; I(A, b) = \mathsf{t} \Rightarrow I(B, c) = \mathsf{t})$

v. $I({\sim} A, a) = \mathsf{t}$ iff $I(A, a^*) = \mathsf{f}$

vi. $I(\Box A, a) = \mathsf{t}$ iff $\forall b \in W(S_\Box ab \Rightarrow I(A, b) = \mathsf{t})$

vii. $I(\Diamond A, a) = \mathsf{t}$ iff $\exists b \in W(S_\Diamond ab \;\&\; I(A, b) = \mathsf{t})$.

For a non-negative integer n, binary relations $S^n_\blacksquare$ and $S^n_\blacklozenge$, respectively associated with $\blacksquare^n$ and $\blacklozenge^n$, on W are defined inductively as follows. For all $a, b \in W$:

(i) $S^0_\blacksquare ab$ iff $a \leq b$

(ii) for $n > 0$,

$$S^n_\blacksquare ab \;\text{ iff }\; \begin{cases} \exists c \in W(S_\Box ac \;\&\; S^{n-1}_\blacksquare cb), & \text{if } \blacksquare^n \text{ denotes } \Box\blacksquare^{n-1} \\ \exists c \in W(S_\boxdot ac \;\&\; S^{n-1}_\blacksquare cb), & \text{if } \blacksquare^n \text{ denotes } \boxdot\blacksquare^{n-1} \end{cases}$$

(iii) $S^0_\blacklozenge ab$ iff $b \leq a$

(iv) for $n > 0$,

$$S^n_\blacklozenge ab \;\text{ iff }\; \begin{cases} \exists c \in W(S_\Diamond ac \;\&\; S^{n-1}_\blacklozenge cb), & \text{if } \blacklozenge^n \text{ denotes } \Box\blacklozenge^{n-1} \\ \exists c \in W(S_{\Diamonddot} ac \;\&\; S^{n-1}_\blacklozenge cb), & \text{if } \blacklozenge^n \text{ denotes } \Diamonddot\blacklozenge^{n-1} \end{cases}$$

Note that (p6) and (p8), which are regarded as extra postulates in general, justify the definition for $n = 1$. Then it is easy to see that for any non-negative integer n,

(a) $I(\blacksquare^n A, a) = \mathsf{t}$ iff $\forall b \in W(S^n_\blacksquare ab \Rightarrow I(A, b) = \mathsf{t})$

(b) $I(\blacklozenge^n A, a) = \mathsf{t}$ iff $\exists b \in W(S^n_\blacklozenge ab \;\&\; I(A, b) = \mathsf{t})$.

Let $A \in \mathsf{Wff}$. Then we say (a) A *holds in* a $\mathbf{B.C}_{\Box\Diamond}$-model $\mathfrak{M} = \langle O, W, R, S_\Box, S_\Diamond, ^*, v\rangle$ iff $I(A, a) = \mathsf{t}$ for every $a \in O$, and (b) A is *valid* in a $\mathbf{B.C}_{\Box\Diamond}$-frame $\mathfrak{F} = \langle O, W, R, S_\Box, S_\Diamond, ^* \rangle$ iff A holds in every $\mathbf{B.C}_{\Box\Diamond}$-model on $\mathfrak{F}$.

All axioms and rules of inference we mentioned have their corresponding *frame conditions*, that is, if a logic $\mathbf{L}$ contains an axiom or a rule of inference, then any $\mathbf{L}$-*frame* $\mathfrak{F} = \langle O, W, R, S_\Box, S_\Diamond, ^* \rangle$ satisfies certain postulates. Here is a list of such correspondences, where $a, b, c, d, e \in W$.

(A10) $Rabc \Rightarrow Rac^*b^*$

(A11) $Rabc \Rightarrow \exists x \in W(Rabx \ \& \ Raxc)$

(A12) $Rabc \ \& \ Rcde \Rightarrow \exists x \in W(Radx \ \& \ Rbxe)$

(A13) $Rabc \ \& \ Rcde \Rightarrow \exists x \in W(Rbdx \ \& \ Raxe)$

(A14) $Rabc \Rightarrow Rbac$

(A15) Raa^*a

(A16) $Rabc \Rightarrow \exists x \in W(Rabx \ \& \ Rxbc)$

(A17) $a \in O \Rightarrow a^* \leq a$

(A18) $Raaa$

(R5) $\exists x \in O(Raxa)$

(DR1) $a \in O \Rightarrow Raaa$

(DR2) $a \in O \ \& \ Rabe \ \& \ Recd \Rightarrow$
$\exists x \in W \exists y \in W(Racx \ \& \ Rbxy \ \& \ Rayd)$

(DR3) $a \in O \ \& \ Rabc \Rightarrow Rac^*b^*$

(DR4) $a \in O \Rightarrow Ra^*aa^*$

(DR5) $a \in O \Rightarrow Raa^*a$

(A21) $Rabc \ \& \ S_\Box cd \Rightarrow \exists x \in W \exists y \in W(S_\Box ax \ \& \ S_\Box by \ \& \ Rxyd)$

(A22) $Rabc \ \& \ S_\Diamond bd \Rightarrow \exists x \in W \exists y \in W(S_\Box ax \ \& \ S_\Diamond cy \ \& \ Rxdy)$

(GA(k, l, m, n)) $S^k_\blacklozenge ab \ \& \ S^m_\blacksquare ac \Rightarrow \exists x \in W(S^l_\blacksquare bx \ \& \ S^n_\blacklozenge cx)$

(R8) $a \in O \ \& \ S_\Box ab \Rightarrow b \in O$

(DR6) $a \in O \ \& \ Rabc \ \& \ S_\Box cd \Rightarrow \exists x \in W(Raxd \ \& \ S_\Box bx)$

(DR7) $a \in O \ \& \ Rabc \ \& \ S_\Diamond bd \Rightarrow \exists x \in W(Radx \ \& \ S_\Diamond cx)$

(DR8) $a \in O \ \& \ S_\Box ab \Rightarrow a \leq b$

(GDR(k, l, m, n)) $a \in O \ \& \ S^k_\blacklozenge ab \ \& \ S^m_\blacksquare ac \Rightarrow \exists x \in W(S^l_\blacksquare bx \ \& \ S^n_\blacklozenge cx)$

L-*models* (on **L**-frames) are defined similarly to $\mathbf{B.C}_{\Box\Diamond}$-models. Soundness is proved in the usual way.

Proposition 1. *Let* **L** *be any logic defined above and* $A \in \mathsf{Wff}$. *If* A *is a theorem of* **L**, *then* A *is valid in every* **L**-*frame.*

4. Completeness

A proof of completeness for (non-modal) relevant logics of the form **L1** and **L2** has been given in Section 4 of [5]. Completeness of relevant modal logics of the form **L3** has been proved in [6]. (Note that **L3** contain no disjunctive rules.) We will prove completeness for relevant modal logics of the form **L4**. Our proof of completeness for **L4** follows the method from Section 4.8 of [5]. It differs somewhat from that for **L3** in the definition of derivable relations.

A comment on the method of proof is in order here. Namely, it seems that simply transferring the method from [5] cannot be successful. This is because JL-theories used there must satisfy closure conditions corresponding to disjunctive rules. To illustrate the problem, let **L** be $\mathbf{DL} \oplus \{(\mathrm{DR1})\}$. Proceeding with the completeness proof along the lines of [5], we must show the following: for a prime **L**-theory Σ, if $B \to C \notin \Sigma$, then there exist prime **L**-theories Γ and Δ such that $R\Sigma\Gamma\Delta$, $B \in \Gamma$ and $C \notin \Delta$. Proving this requires in turn that $\Gamma = \{A \mid B \to A \text{ is a theorem of } \mathbf{L}\}$ should be an **L**-theory. Then, since **L**-theories correspond to JL-theories, according to the definition from [5], we must verify the following: (i) if $A_1, A_2 \in \Gamma$, then $A_1 \wedge A_2 \in \Gamma$; (ii) if $A_1 \to A_2$ is a theorem of **L** and $A_1 \in \Gamma$, then $A_2 \in \Gamma$; (iii) if $C \vee (A_1 \to A_2), C \vee A_1 \in \Gamma$, then $C \vee A_2 \in \Gamma$. It is however not clear how to verify (iii), known as closure condition, or indeed, whether (iii) holds at all. Fortunately, if the closure conditions corresponding to disjunctive rules are restricted to regular **L**-theories, then the proof can be carried out and it turns out that this suffices for completeness.

Our proof hinges on introducing appropriate semantic notions for modalities, having done that all proofs become rather standard, so we will omit almost all of them. Below we list notions essential for proving completeness, with little or no comment. Note however that some of them are rather different than those from [5].

- $\mathbf{L4} \vdash \Psi \to \Phi$ iff there exist $A_1, \cdots, A_m \in \Psi$ ($m \geq 1$) and $B_1, \cdots, B_n \in \Phi$ ($n \geq 1$) such that $A_1 \wedge \cdots \wedge A_m \to B_1 \vee \cdots \vee B_n$ is a theorem of **L4**. $\mathbf{L4} \nvdash \Psi \to \Phi$ denotes that $\mathbf{L4} \vdash \Psi \to \Phi$ does not hold.
- (Ψ, Φ) is **L4**-*pair* iff (a) $\mathbf{L4} \nvdash \Psi \to \Phi$ and (b) $\Psi \cup \Phi = \mathsf{Wff}$.
- $A \vdash_{\mathbf{L4}} B$ iff B is derivable from A by successive applications of the following:
  - Adjunction (i.e., (R2))
  - Provable Implication (i.e., from A infer B, for a provable $A \to B$)
  - Adopted disjunctive rules (DR1)–(DR8) and (GDR(k, l, m, n)).
- $\Psi \vdash_{\mathbf{L4}} \Phi$ iff there exist $A_1, \cdots, A_m \in \Psi$ ($m \geq 1$) and $B_1, \cdots, B_n \in \Phi$ ($n \geq 1$) such that $A_1 \wedge \cdots \wedge A_m \vdash_{\mathbf{L4}} B_1 \vee \cdots \vee B_n$. $\Psi \nvdash_{\mathbf{L4}} \Phi$ denotes that $\Psi \vdash_{\mathbf{L4}} \Phi$ does not hold.
- $\langle \Psi, \Phi \rangle$ is **L4**-*d-pair* iff (a) $\Psi \nvdash_{\mathbf{L4}} \Phi$ and (b) $\Psi \cup \Phi = \mathsf{Wff}$.
- Σ is an **L4**-*theory* iff (a) if $A \in \Sigma$ and $B \in \Sigma$ then $A \wedge B \in \Sigma$, and (b) if $A \to B$ is a theorem of **L4** and $A \in \Sigma$ then $B \in \Sigma$.
- An **L4**-theory Σ is *regular* iff (a) Σ contains all theorems of **L4**, and (b) Σ conforms to whichever of the following closure conditions match disjunctive rules of **L4**:

(DR1) $C \vee A \in \Sigma \ \&\ C \vee (A \to B) \in \Sigma \Rightarrow C \vee B \in \Sigma$

(DR2) $E \vee (A \to B) \in \Sigma \ \&\ E \vee (C \to D) \in \Sigma \Rightarrow$
$E \vee ((B \to C) \to (A \to D)) \in \Sigma$

(DR3) $C \vee (A \to\sim B) \in \Sigma \Rightarrow C \vee (B \to\sim A) \in \Sigma$

(DR4) $C \vee A \in \Sigma \Rightarrow C \vee \sim (A \to\sim A) \in \Sigma$

(DR5) $C \vee (\sim A \to A) \in \Sigma \Rightarrow C \vee A \in \Sigma$

(DR6) $C \vee (A \to B) \in \Sigma \Rightarrow C \vee (\Box A \to \Box B) \in \Sigma$

(DR7) $C \vee (A \to B) \in \Sigma \Rightarrow C \vee (\Diamond A \to \Diamond B) \in \Sigma$

(DR8) $C \vee A \in \Sigma \Rightarrow C \vee \Box A \in \Sigma$

(GDR(k, l, m, n)) $C \vee \blacklozenge^k \blacksquare^l A \in \Sigma \Rightarrow C \vee \blacksquare^m \blacklozenge^n A \in \Sigma$.

- An **L4**-theory Σ is *prime* iff $A \vee B \in \Sigma$ implies either $A \in \Sigma$ or $B \in \Sigma$ (or both).

- Let $\mathsf{Th}(\mathbf{L4})$ be the set of all **L4**-theories. Then a ternary relation R on $\mathsf{Th}(\mathbf{L4})$, and binary relations $S_\Box$ and $S_\Diamond$ on $\mathsf{Th}(\mathbf{L4})$, are defined by

$$\begin{array}{rcl} R\Sigma\Gamma\Delta & \text{iff} & \forall A, B \in \mathsf{Wff}(A \to B \in \Sigma \ \&\ A \in \Gamma \Rightarrow B \in \Delta) \\ S_\Box\Sigma\Gamma & \text{iff} & \forall A \in \mathsf{Wff}(\Box A \in \Sigma \Rightarrow A \in \Gamma) \\ S_\Diamond\Sigma\Gamma & \text{iff} & \forall A \in \mathsf{Wff}(A \in \Gamma \Rightarrow \Diamond A \in \Sigma) \end{array}$$

  for all $\Sigma, \Gamma, \Delta \in \mathsf{Th}(\mathbf{L4})$.

The following proposition has been proved in Section 4.8 of [5]. Note that the notions of **L4**-pairs and **L4**-d-pairs are essentially used in proofs of 2 and 4, respectively. Further, in the proof of 3 the claim "$A \vdash_{\mathbf{L4}} B$ implies $C \vee A \vdash_{\mathbf{L4}} C \vee B$" is used.

Proposition 2.

1. *If* **L4** $\nvdash \Psi \to \Phi$*, then there exists an* **L4**-*pair* (Ψ', Φ') *such that* $\Psi \subseteq \Psi'$ *and* $\Phi \subseteq \Phi'$*. Hence* Ψ' *is a prime* **L4**-*theory.*

2. *Where* Ψ *is a* **L4**-*theory and* Φ *is a set of* Wff *satisfying that* $\Psi \cap \Phi = \emptyset$ *and that closed under disjunction, then there exists a prime* **L4**-*theory* Ψ' *such that* $\Psi \subseteq \Psi'$ *and* $\Psi' \cap \Phi = \emptyset$.

3. *If* $\Psi \nvdash_{\mathbf{L4}} \Phi$*, then there exists an* **L4**-*d-pair* $\langle \Psi', \Phi' \rangle$ *such that* $\Psi \subseteq \Psi'$ *and* $\Phi \subseteq \Phi'$*. Hence* Ψ' *is a prime* **L4**-*theory.*

4. *If* A *is not a theorem of* **L4***, then there exists a regular prime* **L4**-*theory* Π *such that* $A \notin \Pi$.

5. *Suppose that* Σ *and* Γ *are* **L4**-*theories and* Δ *is a prime* **L4**-*theory such that* $R\Sigma\Gamma\Delta$*. Then there exists prime* **L4**-*theories* Σ' *and* Γ' *such that* $\Sigma \subseteq \Sigma'$, $\Gamma \subseteq \Gamma'$ *and* $R\Sigma'\Gamma'\Delta$.

The *canonical* **L4**-*model* $\langle O_c, W_c, R_c, S_{\Box c}, S_{\Diamond c}, g_c, v_c \rangle$ is defined as follows:

- W_c is the set of all prime **L4**-theories

- O_c is the set of all regular prime **L4**-theories
- R_c is the ternary relation R restricted to W_c
- $S_{\Box c}$ is the binary relation $S_{\Box}$ restricted to W_c
- $S_{\Diamond c}$ is the binary relation $S_{\Diamond}$ restricted to W_c
- g_c is the unary operation on W_c defined by $g_c(\Sigma) = \{A \mid\ \sim A \notin \Sigma\}$ for all $\Sigma \in W_c$
- v_c is defined by $v_c(p, \Sigma) = \mathsf{t}$ iff $p \in \Sigma$, for all $p \in \mathsf{Prop}$ and $\Sigma \in W_c$.

The frame underlying the canonical model, i.e., the frame $\langle O_c, W_c, R_c, S_{\Box c}, S_{\Diamond c}, g_c\rangle$ is called the *canonical* **L4**-*frame*. Note that $\leq_c$, $S_{\boxdot c}$, $S_{\diamonddot c}$, $S_{\blacksquare c}{}^n$ and $S_{\blacklozenge c}{}^n$ are defined as in Section 3, and that $\leq_c$ is the set-theoretic inclusion $\subseteq$. We can easily show that v_c is a valuation, and the interpretation I_c associated with v_c satisfies the definition of interpretation.

Lemma 3. *Let $\langle O_c, W_c, R_c, S_{\Box c}, S_{\Diamond c}, g_c\rangle$ be the canonical* **L4**-*frame. For any non-negative integer n and all $\Sigma, \Gamma \in W_c$:*

1. $S_{\blacksquare c}{}^n \Sigma\Gamma$ *iff* $\forall A \in \mathsf{Wff}(\blacksquare^n A \in \Sigma \Rightarrow A \in \Gamma)$
2. $S_{\blacklozenge c}{}^n \Sigma\Gamma$ *iff* $\forall A \in \mathsf{Wff}(A \in \Gamma \Rightarrow \blacklozenge^n A \in \Sigma)$.

Proof. We only prove 1. The proof proceeds by induction on n. The 'only if' part is easy, so we prove the 'if' part. For $n = 0$, it is obvious. For $n > 0$, let $\blacksquare^n$ be either $\Box\blacksquare^{n-1}$ or $\boxdot\blacksquare^{n-1}$. Since the proof for each case is essentially same, we give a proof for the former only. Take $\Psi = \{A \mid \Box A \in \Sigma\}$ and $\Phi = \{\blacksquare^{n-1} A \mid A \notin \Gamma\}$. To show that $\mathbf{L4} \not\vdash \Psi \to \Phi$, suppose otherwise. Then there exist $\Box A_1, \cdots, \Box A_k \in \Sigma$ and $B_1, \cdots, B_l \notin \Gamma$ such that

$$A_1 \wedge \cdots \wedge A_k \to \blacksquare^{n-1} B_1 \vee \cdots \vee \blacksquare^{n-1} B_l$$

is a theorem of **L4**. Since $\blacksquare^{n-1} B_1 \vee \cdots \vee \blacksquare^{n-1} B_l \to \blacksquare^{n-1}(B_1 \vee \cdots \vee B_l)$ is a theorem of **L4**,

$$\Box A_1 \wedge \cdots \wedge \Box A_k \to \Box\blacksquare^{n-1}(B_1 \vee \cdots \vee B_l)$$

is also a theorem of **L4**. Further, since $\Box A_1 \wedge \cdots \wedge \Box A_k \in \Sigma$, we have $\Box\blacksquare^{n-1}(B_1 \vee \cdots \vee B_l) \in \Sigma$. And since $B_1 \vee \cdots \vee B_l \notin \Gamma$, we have $\Box\blacksquare^{n-1}(B_1 \vee$

$\cdots \vee B_l) \notin \Sigma$ by the assumption. Thus we obtain a contradiction, so $\mathbf{L4} \not\vdash \Psi \to \Phi$.

By 1 of Proposition 2, there exists an **L4**-pair (Ψ', Φ') such that $\Psi \subseteq \Psi'$ and $\Phi \subseteq \Phi'$. Then it is easy to show that $\Box A \in \Sigma$ implies $A \in \Psi'$ and that $\blacksquare^{n-1} A \in \Psi'$ implies $A \in \Gamma$. By the hypothesis of induction, there exists $\Psi' \in W_c$ such that $S_{\Box c}\Sigma\Psi'$ and $S_{\blacksquare c}{}^{n-1}\Psi'\Gamma$. □

Lemma 4. *The canonical* **L4***-frame is an* **L4***-frame.*

Proof. Here we only give a proof for the frame conditions corresponding to disjunctive rules (DR6), (DR7), (DR8) and (GDR(k, l, m, n)). For other conditions, see e.g., [5] and [6].

(DR6) Suppose that $R_c\Pi\Sigma\Gamma$ and $S_{\Box c}\Gamma\Delta$ for any $\Pi \in O_c$ and $\Sigma, \Gamma, \Delta \in W_c$. Let $\Psi = \{A \mid \Box A \in \Sigma\}$. First we check that Ψ is an **L4**-theory. Suppose that $A, B \in \Psi$. Then $\Box A, \Box B \in \Sigma$, and hence $\Box(A \wedge B) \in \Sigma$ since $\Sigma \in W_c$. Thus $A \wedge B \in \Psi$. Further, suppose that $A \to B$ is a theorem of **L4** and $A \in \Psi$. Then $\Box A \to \Box B$ is a theorem of **L4** and $\Box A \in \Sigma$. Since $\Sigma \in W_c$, we have $\Box B \in \Sigma$, and hence $B \in \Psi$. Therefore Ψ is an **L4**-theory satisfying $S_{\Box}\Sigma\Psi$.

Next suppose that $A \to B \in \Pi$ and $A \in \Psi$. Then $(\Box A \to \Box B) \vee (A \to B) \in \Pi$ and $\Box A \in \Sigma$. Since $\Pi \in O_c$, we have $(\Box A \to \Box B) \vee (\Box A \to \Box B) \in \Pi$, and hence $\Box A \to \Box B \in \Pi$. By the assumption, we have $\Box B \in \Gamma$, so $B \in \Delta$. Thus we obtain that $R\Pi\Psi\Delta$. By 5 of Proposition 2, there exists $\Psi' \in W_c$ such that $\Psi \subseteq \Psi'$ and $R\Pi\Psi'\Delta$. Therefore there exists $\Psi' \in W_c$ such that $R_c\Pi\Psi'\Delta$ and $S_{\Box c}\Sigma\Psi'$.

(DR7) Suppose that $R_c\Pi\Sigma\Gamma$ and $S_{\Diamond c}\Sigma\Delta$ for any $\Pi \in O_c$ and $\Sigma, \Gamma, \Delta \in W_c$. Let $\Psi = \{A \mid \exists B \in \Delta(B \to A \in \Pi)\}$ and $\Phi = \{A \mid \Diamond A \notin \Gamma\}$. First we check that Ψ is an **L4**-theory. Suppose that $A_1, A_2 \in \Psi$. Then there exist $B_1, B_2 \in \Delta$ such that $B_1 \to A_1, B_2 \to A_2 \in \Pi$. It is easy to see that $B_1 \wedge B_2 \in \Delta$ and $B_1 \wedge B_2 \to A_1 \wedge A_2 \in \Pi$, so we have $A_1 \wedge A_2 \in \Psi$. Further, suppose that $A_1 \to A_2$ is a theorem of **L4** and $A_1 \in \Psi$. Then $(B \to A_1) \to (B \to A_2)$ is a theorem of **L4** and there exists $B \in \Delta$ such that $B \to A_1 \in \Pi$. We have $B \to A_2 \in \Pi$, and hence $A_2 \in \Psi$. Thus Ψ is an **L4**-theory satisfying $R\Pi\Delta\Psi$.

Next suppose that $A, B \in \Phi$. Then $\Diamond A, \Diamond B \notin \Gamma$, and hence $\Diamond(A \vee B) \notin \Gamma$ since $\Gamma \in W_c$. Thus $A \vee B \in \Phi$. Therefore, Φ is closed under disjunction. To show that $\Psi \cap \Phi = \emptyset$, suppose otherwise. Then, $\Diamond A \notin \Gamma$ for some $A \in \Psi \cap \Phi$, and thus there exists $B \in \Delta$ such that $B \to A \in \Pi$. Since $\Pi \in O_c$, we have $(\Diamond B \to \Diamond A) \vee (B \to A) \in \Pi$, and hence $(\Diamond B \to \Diamond A) \vee (\Diamond B \to \Diamond A) \in \Pi$, so $\Diamond B \to \Diamond A \in \Pi$. Further, since $S_{\Diamond c}\Sigma\Delta$, we get $\Diamond B \in \Sigma$. Since $R_c\Pi\Sigma\Gamma$, we have $\Diamond A \in \Gamma$. This is a contradiction.

By 2 of Proposition 2, there exists $\Psi' \in W_c$ such that $\Psi \subseteq \Psi'$ and $\Psi' \cap \Phi = \emptyset$. Taking any $A \in \Psi'$, we have $A \notin \Phi$, and hence $\Diamond A \in \Gamma$. Thus we have $S_{\Diamond}\Gamma\Psi'$. Therefore there exists $\Psi' \in W_c$ such that $R_c\Pi\Delta\Psi'$ and $S_{\Diamond c}\Gamma\Psi'$. This ends the proof of (DR7).

(DR8) Suppose that $S_{\Box c}\Pi\Sigma$ and $A \in \Pi$ for any $\Pi \in O_c$ and $\Sigma \in W_c$. Since $\Pi \in O_c$, we have $\Box A \vee A \in \Pi$, so $\Box A \vee \Box A \in \Pi$, and hence $\Box A \in \Pi$. Thus $A \in \Sigma$, which is the desired result.

(GDR(k, l, m, n)) Suppose that $S_{\blacklozenge c}{}^{k}\Pi\Sigma$ and $S_{\blacksquare c}{}^{m}\Pi\Gamma$ for any $\Pi \in O_c$ and $\Sigma, \Gamma \in W_c$. Let $\Psi = \{A \mid \blacksquare^l A \in \Sigma\}$ and $\Phi = \{A \mid \blacklozenge^n A \notin \Gamma\}$. As in the proof of (DR6), Ψ is an **L4**-theory; as in the proof of (DR7), Φ is closed under disjunction. To show that $\Psi \cap \Phi = \emptyset$, suppose otherwise. Then, for some $A \in \Psi \cap \Phi$, we have $\blacksquare^l A \in \Sigma$ and $\blacklozenge^n A \notin \Gamma$. By assumption, $\blacklozenge^k \blacksquare^l A \in \Pi$ and $\blacksquare^m \blacklozenge^n A \notin \Pi$. Since $\Pi \in O_c$, we have $\blacksquare^m \blacklozenge^n A \vee \blacklozenge^k \blacksquare^l A \in \Pi$, and hence $\blacksquare^m \blacklozenge^n A \vee \blacksquare^m \blacklozenge^n A \in \Pi$, so $\blacksquare^m \blacklozenge^n A \in \Pi$. This is a contradiction.

Again, by 2 of Proposition 2, there exists $\Psi' \in W_c$ such that $\Psi \subseteq \Psi'$ and $\Psi' \cap \Phi = \emptyset$. Further, taking $A \in \Psi'$, we have $A \notin \Phi$, and hence $\blacklozenge^n A \in \Gamma$. Thus $S^n_{\blacklozenge}\Gamma\Psi'$. Therefore, there exists $\Psi' \in W_c$ such that $S_{\blacksquare c}{}^{l}\Sigma\Psi'$ and $S_{\blacklozenge c}{}^{n}\Gamma\Psi'$. $\Box$

The following lemma is proved by induction on the length of A. For non-modal cases, see e.g., [5], and for modal cases, see [6].

Lemma 5. *Let* $\langle O_c, W_c, R_c, S_{\Box c}, S_{\Diamond c}, g_c, v_c \rangle$ *be the canonical* **L4***-model. For all* $A \in \mathsf{Wff}$ *and* $\Sigma \in W_c$*,* $I_c(A, \Sigma) = \mathsf{t}$ *iff* $A \in \Sigma$*.*

Finally, we obtain the generic completeness theorem for **L4**.

Theorem 6. *Let* $A \in \mathsf{Wff}$*. Then* A *is a theorem of* **L4** *iff* A *is valid in every* **L4***-frame.*

Acknowledgments. The author would like to thank Professor Robert K. Meyer for his suggestion of this subject. The author also grateful to Dr. Tomasz Kowalski and Dr. Yoshihito Tanaka for their many valuable comments and encouragement. Finally, the author acknowledges the anonymous referee for her/his helpful suggestions.

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