REPORTS ON MATHEMATICAL LOGIC
44 (2009), 85–103

Leonardo Manuel CABRER

NON CANONICITY OF BL-ALGEBRAS

A b s t r a c t. In this work we study canoncity of **BL**-algebras and some of its subvarieties. We prove that the only subvarieties of **ŁΠG** (see page 94) that are σ-canonical and π-canonical are the ones generated by finite families of finite algebras.

1. Introduction and Preliminaries

Canonical extensions were introduced by Jónsson and Tarski for Boolean algebras with operators (see [12],[13]) and generalized for distributive lattices, lattices, and posets with different internal operations in [4],[5],[6],[7].

Given a distributive lattice $\boldsymbol{L} = \langle L, \wedge, \vee \rangle$, an **extension** is a one to one lattice homomorphism $e : L \to M$. To simplify notation, we usually suppress the embedding e and we call $\boldsymbol{M}$ an extension of $\boldsymbol{L}$ assuming that $\boldsymbol{L}$ is a sublattice of $\boldsymbol{M}$.

An extension $\boldsymbol{M}$ is called a **completion** if $\boldsymbol{M}$ is a complete lattice. Given an extension $\boldsymbol{M}$ of $\boldsymbol{L}$, $\boldsymbol{L}$ is said to be **separating** in $\boldsymbol{M}$ if for every completely join irreducible element $u \in M$ and every completely meet

Received 26 December 2006
Dedicated to the memory of Professor Mario Joaquín Inza

irreducible element $p \in M$ satisfying $p \leq u$ there exist $a \in L$ such that $p \leq a \leq u$. Given a completion $\boldsymbol{M}$ of $\boldsymbol{L}$, $\boldsymbol{L}$ is called **compact** if for every $X, Y \subseteq L$ satisfying $\bigwedge X \leq \bigvee Y$ there exist finite sets $F \subseteq X$ and $G \subseteq Y$ such that $\bigwedge F \leq \bigvee G$.

The **canonical extension** $\boldsymbol{L}^\sigma$ of a bounded distributive lattice $\boldsymbol{L}$ is a doubly algebraic distributive lattice that contains $\boldsymbol{L}$ as a separating and compact sublattice. The canonical extension if unique up to lattice isomorphisms.

An element $p \in \boldsymbol{L}^\sigma$ is called closed if there exists $X \subseteq L$ such that $p = \bigwedge X$. The set of closed elements of $\boldsymbol{L}^\sigma$ is noted by $K(\boldsymbol{L}^\sigma)$. The notion of open elements is defined dually, and the set of open elements is denoted $O(\boldsymbol{L}^\sigma)$

There are two rival extensions for an isotone function $f : \boldsymbol{L} \to \boldsymbol{M}$: one is called the canonical extension and noted $f^\sigma : L^\sigma \to M^\sigma$ and the other is called the dual canonical extension and noted $f^\pi : L^\sigma \to M^\sigma$. There are many ways to define of these extensions (see [5],[6]). For further use, we define them according to the next Lemma.

Lemma 1 (See [5]). *Let $f : \boldsymbol{L} \to \boldsymbol{M}$ be an isotone function from the distributive lattice $\boldsymbol{L}$ into $\boldsymbol{M}$, then*

1. $f^\sigma(p) = \bigwedge \{f(a) : p \leq a \text{ and } a \in L\}$, *for all* $p \in K(\boldsymbol{L}^\sigma)$,
2. $f^\pi(u) = \bigvee \{f(a) : a \leq u \text{ and } a \in L\}$, *for all* $u \in O(\boldsymbol{L}^\sigma)$,
3. $f^\sigma(x) = \bigvee \{f^\sigma(p) : p \leq x \text{ and } p \in K(\boldsymbol{L}^\sigma)\}$ *for all* $x \in \boldsymbol{L}^\sigma$,
4. $f^\pi(x) = \bigwedge \{f^\pi(u) : x \leq u \text{ and } u \in O(\boldsymbol{L}^\sigma)\}$ *for all* $x \in \boldsymbol{L}^\sigma$

A function $f : \boldsymbol{L} \to \boldsymbol{M}$, is called **smooth** if $f^\sigma = f^\pi$.

Given a distrubutive lattice $\boldsymbol{L}$, $\boldsymbol{L}^d$ denote the distributive lattice with the reversing order. Let $\boldsymbol{L} \times \boldsymbol{M}$ be the product of two distributive lattices. The proof of the following results can be found in [6].

1. $\left(\boldsymbol{L}^d\right)^\sigma = (\boldsymbol{L}^\sigma)^d$
2. $K\left(\left(\boldsymbol{L}^d\right)^\sigma\right) = O((\boldsymbol{L}^\sigma))$ and $O\left(\left(\boldsymbol{L}^d\right)^\sigma\right) = K((\boldsymbol{L}^\sigma))$.
3. $(\boldsymbol{L} \times \boldsymbol{M})^\sigma = \boldsymbol{L}^\sigma \times \boldsymbol{M}^\sigma$

4. $K(\boldsymbol{L} \times \boldsymbol{M})^{\sigma} = K(\boldsymbol{L}^{\sigma}) \times K(\boldsymbol{M}^{\sigma})$ and $K(\boldsymbol{L} \times \boldsymbol{M})^{\sigma} = K(\boldsymbol{L}^{\sigma}) \times K(\boldsymbol{M}^{\sigma})$.

Using the previous observations the extensions $f^{\sigma}, f^{\pi}\colon \left(\prod_{i=1}^{n} \boldsymbol{L}_i\right)^{\sigma} \to \boldsymbol{M}^{\sigma}$ of a function $f : \prod_{i=1}^{n} \boldsymbol{L}_i \to \boldsymbol{M}$ that is isotone or antitone in each coordinate, can be computed according to Lemma 1.

Given an algebra $\boldsymbol{A} = \left\langle A, \wedge, \vee, \{f_i\}_{i \in I}\right\rangle$ with a distributive lattice reduct such that each operation f_i is isotone or antitone in each coordinate, this allows us to define two candidates to extend an algebra $\boldsymbol{A}$ with a lattice reduct, $\boldsymbol{A}^{\sigma} = \left\langle A^{\sigma}, \wedge, \vee, \{f_i^{\sigma}\}_{i \in I}\right\rangle$ and $\boldsymbol{A}^{\pi} = \left\langle A^{\sigma}, \wedge, \vee, \{f_i^{\pi}\}_{i \in I}\right\rangle$ called the canonical and the dual canonical extensions of $\boldsymbol{A}$. A class of algebras is σ-canonical or π-canonical if it is closed under canonical or dual canonical extensions respectively.

BL-algebras were introduced by Hájek (see [10]) as the algebraic counterpart of basic logic. A **BL**-algebra $\boldsymbol{A} = \langle A, \odot, \to, \wedge, \vee, 0, 1\rangle$ is a bounded commutative integral residuated lattice that satisfies prelinearity and divisibility. Prelinearity is given by the equation

$$(a \to b) \vee (b \to a) = 1,$$

and it implies that the variety of **BL**-algebras is generated by totally ordered **BL**-algebras. Divisibility is given by the equation:

$$x \odot (x \to y) = y \odot (y \to x).$$

For a precise definition and general results on residuated lattices see [11] or [15].

BL-algebras form a variety or equational class. They can be viewed as bounded distributive lattices with two operators: a product $\odot$ and an implication $\to$. The product is isotone in both coordinates and the implication is antitone in the first coordinate and isotone in the second one (monotone operators in the sense of [4]). Therefore one can analyze the canonicity of this variety and its subvarieties.

For general references and results of **BL**-algebras see [10],[16].

There are many results, positive and negative about canonicity of different classes of algebras. One of the strongest is the one given in [6] where it is proved that if a class $\mathfrak{K}$ is closed under ultraproducts and σ-canonical

(π-canonical) extensions then the variety generated by $\mathfrak{K}$ is σ-canonical (π-canonical). In [8], Gerhke and Priestley have proved that the only subvarieties of **MV**-algebras that are canonical are exactly the finitely generated ones. In this work we will study the canonicity of **BL**-algebras and some of its subvarieties. We will see that **BL**-algebras are neither σ-canonical nor π-canonical and the only subvarieties generated by single component chains (see page 94) that are σ-canonical are finitely generated.

2. Non canonicity of BL-algebras

In this section we prove that the variety of **BL**-algebras is neither σ-canonical nor π-canonical checking that the canonical and dual canonical extension of the Chang algebra are not **BL**-algebras.

Let us recall that the Chang algebra is defined as $\boldsymbol{C} = \langle C, \wedge, \vee, \odot, \rightarrow, 0, 1\rangle$ where $C = \{(0, a) : a \in \mathbb{Z}^+\} \cup \{(1, b) : b \in \mathbb{Z}^-\}$. The lattice order of $\boldsymbol{C}$ is the lexicographic order in the product (see Fig. 1). The product and the implication are defined as follows:

$$(i,a) \odot (j,b) = \begin{cases} (0,0) & \text{if } i+j=0, \\ (0, 0 \vee (a+b)) & \text{if } i+j=1, \\ (1, a+b) & \text{if } i+j=2 \end{cases}$$

$$(i,a) \rightarrow (j,b) = \begin{cases} (0, b-a) & \text{if } i=1 \text{ and } j=0, \\ (1, 0 \wedge (b-a)) & \text{if } i=j, \\ (1,0) & \text{if } i=0 \text{ and } j=1. \end{cases}$$

The canonical extension of the lattice reduct of $\boldsymbol{C}$ has just two new points, following the notation of [8] we will call them x and y (see Fig. 1).

Proposition 2. *Let $\odot^\sigma$ and $\odot^\pi$ be the canonical and dual canonical extensions of the operator $\odot$. Since $\odot^\sigma$ and $\odot^\pi$ are commutative the following tables describe them completely*

$$a \odot^\sigma b = \begin{cases} a \odot b & \textit{if } a, b \in C, & \\ (0,0) & \textit{if } a = x \textit{ and } b = (0,i), & (a) \\ x & \textit{if } a = x \textit{ and } b \geq x, & (b) \\ (0,0) & \textit{if } a = x \textit{ and } b = y, & (c) \\ (0,0) & \textit{if } a = y \textit{ and } b \leq y, & (d) \\ y & \textit{if } a = y \textit{ and } b = (1,j), & (e) \end{cases}$$

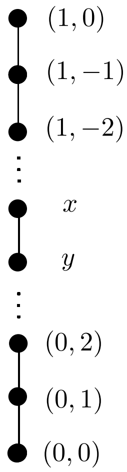

Fig. 1: $\boldsymbol{C}^{\sigma}$

and

$$a \odot^{\pi} b = \begin{cases} y & \text{if } a = x \text{ and } b = y \quad (f) \\ & \text{or if } a = y \text{ and } b = x, \\ a \odot^{\sigma} b & \text{otherwise.} \end{cases}$$

Where $i \in \mathbb{Z}^{+}$ *and* $j \in \mathbb{Z}^{-}$.

Proof. Note that $K(\boldsymbol{C}^{\sigma}) = C \cup \{x\}$ and $O(\boldsymbol{C}^{\sigma}) = C \cup \{y\}$. Thus, $K((\boldsymbol{C} \times \boldsymbol{C})^{\sigma}) = (C \cup \{x\}) \times (C \cup \{x\})$ and $O((\boldsymbol{C} \times \boldsymbol{C})^{\sigma}) = (C \cup \{y\}) \times (C \cup \{y\})$.

We first calculate $\odot^{\sigma}$.

(a) Since $(x, (0, i)) \in K((\boldsymbol{C} \times \boldsymbol{C})^{\sigma})$,

$$\begin{aligned} x \odot^{\sigma} (0, i) &= \bigwedge \{c \odot d : x \leq c \text{ and } (0, i) \leq d \text{ and } c, d \in C\} \\ &= \bigwedge \left\{(1, k) \odot (0, i) : k \in \mathbb{Z}^{-}\right\} \\ &= \bigwedge \left\{(0, 0 \vee (k + i)) : k \in \mathbb{Z}^{-}\right\} \\ &= (0, 0). \end{aligned}$$

Since $(x, (1, j)), (x, x) \in K((\boldsymbol{C} \times \boldsymbol{C})^{\sigma})$, (b) follows in a similar way.

(c)

$$\begin{aligned} x \odot^\sigma y &= \bigvee \{c \odot^\sigma d : c, d \in K(\boldsymbol{C}^\sigma), c \leq x \text{ and } d \leq y\} \\ &= \bigvee \left\{x \odot^\sigma (0,k) : k \in \mathbb{Z}^+\right\} \\ &= (0,0). \end{aligned}$$

(d) Since $\odot^\sigma$ is commutative and monotone in each coordinate, we have that $(0,0) \leq y \odot^\sigma (0,i) \leq y \odot^\sigma y \leq x \odot^\sigma y = (0,0)$

(e)

$$\begin{aligned} y \odot^\sigma (1,j) &= \bigvee \{c \odot^\sigma (1,j) : c \in K(\boldsymbol{C}^\sigma), c \leq y\} \\ &= \bigvee \left\{(0,i) \odot (1,j) : i \in \mathbb{Z}^+\right\} \\ &= \bigvee \left\{(0, 0 \vee (i+j)) : i \in \mathbb{Z}^+\right\} \\ &= y. \end{aligned}$$

Since by [6, Theorem 2.20], $\odot^\sigma$ and $\odot^\pi$ are equal in $K((\boldsymbol{C} \times \boldsymbol{C})^\sigma) \cup O((\boldsymbol{C} \times \boldsymbol{C})^\sigma)$, we only have to prove (f).

$$\begin{aligned} x \odot^\pi y &= \bigwedge \{c \odot^\pi d : c, d \in O(\boldsymbol{C}^\sigma), x \leq c \text{ and } y \leq d\} \\ &= \bigwedge \left\{(1,j) \odot^\pi y : j \in \mathbb{Z}^-\right\} \\ &= y. \end{aligned}$$

□

Proposition 3. *The canonical and dual canonical extensions $\rightarrow^\sigma$ and $\rightarrow^\pi$ of $\rightarrow$ are given in the following tables:*

$$a \rightarrow^\sigma b = \begin{cases} a \rightarrow b & \text{if } a, b \in C, \\ (1,0) & \text{if } a = y \text{ and } b \geq x & (a) \\ x & \text{if } a = y \text{ and } b = (0,i) & (b) \\ (1,0) & \text{if } a = (0,i) \text{ and } b = y & (c) \\ y & \text{if } a \geq x \text{ and } b = y & (d) \\ x & \text{if } a = b = y & (e) \\ (1,0) & \text{if } a = (0,i) \text{ and } b = x & (f) \\ x & \text{if } a = (1,j) \text{ and } b = x & (g) \\ x & \text{if } a = b = x & (h) \\ y & \text{if } a = x \text{ and } b = (0,i) & (i) \\ (1,0) & \text{if } a = x \text{ and } b = (1,j) & (j) \end{cases}$$

and

$$a \to^{\pi} b = \begin{cases} (1,0) & \textit{if } a = b = y \quad \textit{(k)} \\ (1,0) & \textit{if } a = b = x \quad \textit{(l)} \\ a \to^{\sigma} b & \textit{otherwise.} \end{cases}$$

Proof. Note that

$$K\left(\left(\boldsymbol{C}^{d} \times \boldsymbol{C}\right)^{\sigma}\right) = (C \cup \{y\}) \times (C \cup \{x\})$$

and

$$O\left(\left(\boldsymbol{C}^{d}\right)^{\sigma} \times \boldsymbol{C}^{\sigma}\right) = (C \cup \{x\}) \times (C \cup \{y\}).$$

We first calculate $\to^{\sigma}$.

(a) Since $(y, x) \in K\left(\left(\boldsymbol{C}^{d} \times \boldsymbol{C}\right)^{\sigma}\right)$,

$$\begin{aligned} y \to^{\sigma} x \quad &= \bigwedge \{(0,k) \to (1,l) : \text{ with } k \in \mathbb{Z}^{+} \text{ and } l \in \mathbb{Z}^{-}\} \\ &= (1,0). \end{aligned}$$

The case $(y, (1,j))$ follows in a similar way.

(b) Since $(y, (0,i)) \in K\left(\left(\boldsymbol{C}^{d} \times \boldsymbol{C}\right)^{\sigma}\right)$

$$\begin{aligned} y \to^{\sigma} (0,i) \quad &= \bigwedge \{(0,k) \to (0,i) : \text{ with } k \in \mathbb{Z}^{+}\} \\ &= \bigwedge \{(1, 0 \wedge (i-k)) : \text{ with } k \in \mathbb{Z}^{+}\} \\ &= x. \end{aligned}$$

(c)

$$\begin{aligned} (0,i) \to^{\sigma} y \quad &= \bigvee \{(0,i) \to^{\sigma} (0,k) : k \in \mathbb{Z}^{+}\} \\ &= (0,1). \end{aligned}$$

(d)

$$\begin{aligned} x \to^{\sigma} y \quad &= \bigvee \{(1,l) \to^{\sigma} (0,k) : k \in \mathbb{Z}^{+} \text{ and } l \in \mathbb{Z}^{-}\} \\ &= \bigvee \{(0, k-l) : k \in \mathbb{Z}^{+} \text{ and } l \in \mathbb{Z}^{-}\} \\ &= y. \end{aligned}$$

The case $((1,j), y)$ follows in a similar way.

(e)

$$\begin{aligned} y \to^{\sigma} y \quad &= \bigvee \{y \to^{\sigma} (0,k) : k \in \mathbb{Z}^{+}\} \\ &= x. \end{aligned}$$

(f) Since $((0,i),x) \in K\left(\left(\boldsymbol{C}^d \times \boldsymbol{C}\right)^\sigma\right)$

$$\begin{aligned}(0,i) \to^\sigma x &= \bigwedge \{(0,i) \to (1,k) : k \in \mathbb{Z}^-\}\\ &= (1,0).\end{aligned}$$

(g) Since $((1,j),x) \in K\left(\left(\boldsymbol{C}^d \times \boldsymbol{C}\right)^\sigma\right)$

$$\begin{aligned}(1,j) \to^\sigma x &= \bigwedge \{(1,j) \to (1,k) : k \in \mathbb{Z}^-\}\\ &= \bigwedge \{(1,0 \wedge (k-j)) : k \in \mathbb{Z}^-\}\\ &= x.\end{aligned}$$

(h)

$$\begin{aligned}x \to^\sigma x &= \bigvee \{(1,k) \to^\sigma x : k \in \mathbb{Z}^-\}\\ &= x.\end{aligned}$$

(i)

$$\begin{aligned}x \to^\sigma (0,i) &= \bigvee \{(1,k) \to^\sigma (0,i) : k \in \mathbb{Z}^-\}\\ &= \bigvee \{(0,i-k) : k \in \mathbb{Z}^-\}\\ &= y.\end{aligned}$$

(j)

$$\begin{aligned}x \to^\sigma (1,j) &= \bigvee \{(1,k) \to^\sigma (1,j) : k \in \mathbb{Z}^-\}\\ &= \bigvee \{(1,0 \wedge (j-k)) : k \in \mathbb{Z}^-\}\\ &= (1,0).\end{aligned}$$

Since $\to^\sigma$ and $\to^\pi$ are equal in $K\left(\left(\boldsymbol{C}^d \times \boldsymbol{C}\right)^\sigma\right) \cup O\left(\left(\boldsymbol{C}^d \times \boldsymbol{C}\right)^\sigma\right)$, we only have to prove (k) and (l)

(k)

$$\begin{aligned}y \to^\pi y &= \bigwedge \{(0,k) \to^\pi y : k \in \mathbb{Z}^+\}\\ &= \bigwedge \{(0,k) \to^\sigma y : k \in \mathbb{Z}^+\}\\ &= (1,0).\end{aligned}$$

(l) follows in a similar way. □

Corollary 4. *The product and the implication of $\boldsymbol{C}$ are not smooth operations.*

Theorem 5. *Neither $\boldsymbol{C}^\sigma$ nor $\boldsymbol{C}^\pi$ are* **BL***-algebras.*

Proof. In fact in neither case the equation $a \odot (a \to b) = b \odot (b \to a)$ is satisfied.

For $\boldsymbol{C}^\sigma$ let us consider

$$\begin{aligned} x \odot^\sigma (x \to^\sigma y) &= x \odot^\sigma y \\ &= (0,0)\,, \end{aligned}$$

and

$$\begin{aligned} y \odot^\sigma (y \to^\sigma x) &= x \odot^\sigma (1,0) \\ &= x. \end{aligned}$$

For the case of $\boldsymbol{C}^\pi$ let us consider

$$\begin{aligned} x \odot^\pi (x \to^\pi (0,0)) &= x \odot^\pi y \\ &= y, \end{aligned}$$

and

$$\begin{aligned} (0,0) \odot^\pi ((0,0) \to^\pi x) &= (0,0) \odot^\pi (1,0) \\ &= (0,0)\,. \end{aligned}$$

□

The previous Theorem proves that the variety of **BL**-algebras as well as each subvariety of **BL**-algebras which contain $\boldsymbol{C}$ are neither σ-canonical nor π-canonical. In fact, it is easy to see that neither $\boldsymbol{C}^\sigma$ nor $\boldsymbol{C}^\pi$ satisfies the equation

$$a \to (b \to c) = (a \odot b) \to c.$$

Thus, neither $\boldsymbol{C}^\sigma$ nor $\boldsymbol{C}^\pi$ are residated lattices. This proves that the varieties of Residuated Lattices, **MTL**-algebras and **IMTL**-algebras (see [3]) are neither σ-canonical nor π-canonical. It is worth noting that in [6], Gerhke and Jónsson have proved that if $\boldsymbol{L}$ is a residuated lattice the pair $(\odot^\sigma, \to^\pi)$ is an adjoint pair, which proves that L^π (L^σ) is a residuated lattice if and only if $\odot$ $(\to)$ is a smooth operation. We have proved in Corollary 4, that neither the product nor the implication of the Chang algebra are smooth operation.

3. Canonicity of Subvarieties of BL-algebras

Among the subvarieties of **BL**-algebras it is worth pointing out the varieties **Ł** of **MV**-algebras, **Π** of product algebras, and **G** of Gödel algebras. Let us recall that the algebras in **Ł** are **BL**-algebras satisfying the equation

$$a = (a \to 0) \to 0,$$

the Product algebras are **BL**-algebras satisfying the equations

$$\begin{aligned} 0 = &\quad a \wedge (a \to 0)\,, \\ 1 = &\quad ((c \to 0) \to 0) \to ((a \odot c \to b \odot c) \to (a \to b))\,, \end{aligned}$$

and Gödel algebras are **BL**-algebras satisfying the equation

$$a = a \odot a.$$

It is known that each subvariety of **BL**-algebras is generated by **BL**-chains (linearly ordered **BL**-algebras), see [10]. In [9], it is proved that each **BL**-chain can be embedded in a saturated **BL**-chain and that each saturated **BL**-chain can be decomposed as an ordinal sum of MV, Gödel and Product chains.

Now we focus our attention on the subvariety **ŁΠG** which is the subvariety of **BL**-algebras generated by the family of **MV**-chains, product chains and Gödel chains. The lattice of subvarieties of **ŁΠG** was completely described by Di Nola et al. in [2]. Let us recall an important result of that paper.

Theorem 6 ([2] Theorem 3)**.** *The lattice of the subvarieties of* **ŁΠG** *is the Cartesian product of the lattices of subvarieties of* **Ł**, **Π** *and* **G**.

The previous suggests that in order to study the canonicity of subvarieties of **ŁΠG**, we should analyze the canonicity of the subvarieties of **Ł**, **Π** and **G**.

Ł

The lattice of subvarieties of **Ł** was determined by Komori in [14]. He proved that every proper subvariety of **Ł** has a finite number of generators (see [14, Theorem 4.11]). We will describe the generators in the language of **BL**-algebras. The finite generators are the chains $S_n = \left\{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\right\}$

and the infinite generators are $S_n^\omega = \left\{(x,y) : x \in \left\{\frac{1}{n}, \ldots, \frac{n-1}{n}\right\}, y \in \mathbb{Z}\right\} \cup \left\{(0,y) : y \in \mathbb{Z}^+\right\} \cup \left\{(1,y) : y \in \mathbb{Z}^-\right\}$, where the product is given by

$$(x,y) \odot (z,u) = \begin{cases} (0,0) & \text{if } x+z<1, \\ (0, \max(0, u+y)) & \text{if } x+z=1, \\ (x+z-1, u+y) & \text{if } x+z>1. \end{cases}$$

and the implication is given by

$$(x,y) \to (z,u) = \begin{cases} (1,0) & \text{if } x<z, \\ (1, \min(0, u-y)) & \text{if } x=z, \\ (1-x+z, u-y) & \text{otherwise.} \end{cases}$$

In [8], Gehrke and Priestley have proved that a subvariety of **MV**-algebras is canonical (σ or π) if and only if it is generated by a finite family of finite algebras. This result is based on the fact that the Chang algebra $\boldsymbol{C}$ belongs to each subvariety of **MV**-algebras non finitely generated and they prove that the canonical extension of a Chang algebra is not an **MV**-algebra. In Section 2 we have proved that the canonical and the dual canonical extensions of the Chang algebra seen as **BL**-algebra are not **BL**-algebras. Therefore, each subvariety of **BL**-algebras that contains S_n^ω for some $n \geq 1$ is neither σ-canonical nor π-canonical.

G

The variety **G** is the variety of linear Heyting algebras. It is known that the variety **G** is generated by any infinite Gödel chain. Its proper subvarieties are the varieties $\mathbf{G}_n$, $n \geq 2$, where $\mathbf{G}_n$ is the variety generated by the n elements chain. Clearly $\mathbf{B} = \mathbf{G}_2 \subsetneq \mathbf{G}_3 \subsetneq \ldots \mathbf{G}_n \ldots \subsetneq \mathbf{G}$.

Remark 7. Since each $\mathbf{G}_n$ is generated by a finite algebra we have that $\mathbf{G}_n$ is σ-canonical and π-canonical for each $n \geq 2$.

In [6] Gehrke and Jónsson proved the following result.

Theorem 8. *The variety of Heyting algebras is π-canonical but not σ-canonical.*

The example given by Gehrke and Jónsson to prove that the variety of Heyting algebras is not σ-canonical is a chain. Since the canonical extension of a chain is also a chain, we obtain the following result.

Corollary 9. **G** *is π-canonical but not σ-canonical.*

Π

Cignoli and Torrens have proved in [1, Corollary 2.10] that the only proper subvariety of $\mathbf{\Pi}$ is the variety of Boolean algebras. Now we prove that Π is neither σ-canonical nor π-canonical.

Let $\boldsymbol{P} = \langle [0,1], \min, \max, *, \rightarrow, 0, 1\rangle$ be the interval product algebra, where $*$ is the usual product of numbers and the implication is given by

$$a \rightarrow b = \begin{cases} 1 & \text{if } a \leq b, \\ b/a & \text{if } b < a. \end{cases}$$

Given $\boldsymbol{L}$ a distributive lattice let us denote $X(\boldsymbol{L})$ the set of prime filters of $\boldsymbol{L}$. The canonical extension of a distributive lattice can be constructed as the lattice of increasing subsets of $\langle X(\boldsymbol{L}), \subseteq\rangle$ (see [5]). Using this and the fact that in a chain the prime filters coincide with the set of non empty increasing subsets, the following proposition follows straightforward. We include here an independent proof for the sake of completeness.

Proposition 10. *Consider the lattice*

$$\boldsymbol{L} = ([0,1] \times \{-1,0,1\}) - \{(0,-1),(1,1)\}$$

with the lexicographic order in the product (see Fig. 2). The embedding $e : [0,1] \rightarrow L$ *defined by* $e(a) = (a, 0)$ *is the canonical completion of the lattice* $\langle [0,1], \min, \max, 0, 1\rangle$.

Proof. We will prove that the image of e is a separating compact sublattice of $\boldsymbol{L}$.

Let $(a,i), (b,j) \in L$ be a completely meet irreducible and a completely join irreducible elements respectively, such that $(a,i) \leq (b,j)$. It is easy to see that $i \in \{-1, 0\}$ and $j \in \{0,1\}$. If $a < b$, let $c \in [0,1]$ such that $b < c < a$ thus

$$(a,i) < (c,0) = e(c) < (b,j).$$

If $b = a$ Therefore,

$$(a,i) \leq (a,0) = e(a) \leq (a,j) = (b,j).$$

Thus $\varepsilon([0,1])$ is a separating sublattice of $\boldsymbol{L}$.

Now let A and B be subsets of $[0,1]$ such that $\bigwedge e(A) \leq \bigvee e(B)$. It is easy to see that:

$$\bigwedge e(A) = \begin{cases} e\left(\bigwedge A\right) & \text{if } \bigwedge A \in A \\ \\ \left(\bigwedge A, 1\right) & \text{otherwise.} \end{cases}$$

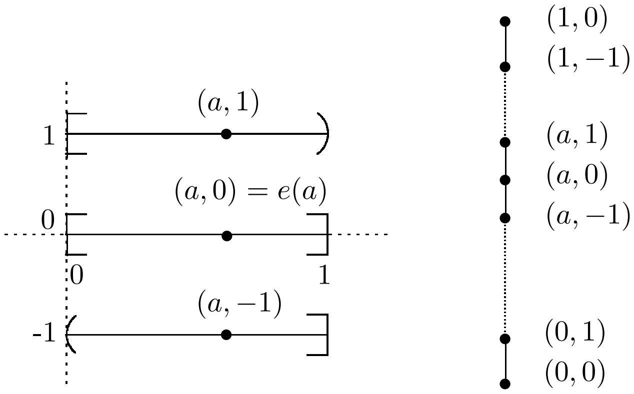

Fig. 2: $\boldsymbol{L}$

and

$$\bigvee e(B) = \begin{cases} e\left(\bigvee B\right) & \text{if } \bigvee B \in B \\ \left(\bigvee B, -1\right) & \text{otherwise.} \end{cases}$$

Suppose that $\bigwedge A \notin A$ and $\bigvee B \notin B$. Thus $\left(\bigwedge A, 1\right) \leq \left(\bigvee B, -1\right)$, i.e., $\bigwedge A < \bigvee B$; it is easy to see that there exist $a \in A$ and $b \in B$ such that $a < b$, and the result follows.

If $\bigwedge A \in A$ or $\bigvee B \in B$ the result follows in a similar way. Thus, $e([0, 1])$ is a compact sublattice of $\boldsymbol{L}$, and the result follows. □

Proposition 11. *Let $\boldsymbol{P}$ be the interval product algebra. Since $*^{\sigma}$ and $*^{\pi}$ are commutative, the following tables describe them completely:*

$$(a, i) *^{\sigma} (b, j) = \begin{cases} (a * b, 0) & \text{if } i = j = 0, & \\ (a * b, 1) & \text{if } i = 0,\ j = 1,\ \text{and } a \neq 0, & (a) \\ (0, 0) & \text{if } i = 0,\ j = 1,\ \text{and } a = 0, & (b) \\ (a * b, 1) & \text{if } i = j = 1, & (c) \\ (a * b, -1) & \text{if } i = -1,\ j = 1,\ \text{and } b \neq 0, & (d) \\ (0, 1) & \text{if } i = -1,\ j = 1,\ \text{and } b = 0, & (e) \\ (a * b, -1) & \text{if } i = -1,\ j = -1, & (f) \\ (a * b, -1) & \text{if } i = -1,\ j = 0,\ \text{and } b \neq 0, & (g) \\ (0, 0) & \text{if } i = -1 \text{ and } (b, j) = (0, 0). & (h) \end{cases}$$

and

$$(a,i) *^{\pi} (b,j) = \begin{cases} (a*b,1) & \textit{if } i=1 \textit{ and } j=-1 \\ & \textit{or if } i=-1 \textit{ and } j=1, \\ (a,i) *^{\sigma} (b,j) & \textit{otherwise.} \end{cases} \quad \textit{(i)}$$

Proof. Note that

$$K\left([0,1]^{\sigma}\right) = \left([0,1]\times\{0\}\right)\cup\left([0,1)\times\{1\}\right)$$

and

$$O\left([0,1]^{\sigma}\right) = \left([0,1]\times\{0\}\right)\cup\left((0,1]\times\{-1\}\right).$$

First we calculate $*^{\sigma}$.

(a) (b) Since $((a,0),(b,1)) \in K\left(([0,1]\times[0,1])^{\sigma}\right)$,

$$\begin{aligned} (a,0) *^{\sigma} (b,1) &= \bigwedge\{(c*d,0) : a \le c \text{ and } b<d\} \\ &= \bigwedge\{(a*d,0) : b<d\}. \end{aligned}$$

If $a=0$, $(a,0) *^{\sigma} (b,1) = (0,0)$. If $a \neq 0$, $(a,0) *^{\sigma} (b,1) = (a*b,1)$.

(c) Since $((a,1),(b,1)) \in K\left(([0,1]\times[0,1])^{\sigma}\right)$ follows as (a).

(d) Since $b \neq 0$,

$$\begin{aligned} (a,-1) *^{\sigma} (b,1) &= \bigvee\{(c,k) *^{\sigma} (b,1) : (c,k) \le (a,-1),\ k\in\{0,1\}\} \\ &= \bigvee\{(c,k) *^{\sigma} (b,1) : c<a,\ k\in\{0,1\}\} \\ &= \bigvee\{(c*b,1) : c<a\}\cup\{(0,0)\} \\ &= (a*b,-1). \end{aligned}$$

(e), (f) and (g) follow as (d).

(h) Since $(b,j)=(0,0)$,

$$\begin{aligned} (a,-1) *^{\sigma} (b,j) &= \bigvee\{(c,k) *^{\sigma} (0,0) : c<a,,\ k\in\{0,1\}\} \\ &= (0,0). \end{aligned}$$

Since by $*^{\sigma}$ and $*^{\pi}$ are equal restricted to

$$K\left(([0,1]\times[0,1])^{\sigma}\right)\cup O\left(([0,1]\times[0,1])^{\sigma}\right),$$

we only have to prove (i)

$$\begin{aligned}(a,1) *^{\pi} (b,-1) &= \bigwedge \{(c,k) *^{\pi} (b,-1) : a < c,\ k \in \{0,-1\}\} \\ &= \bigwedge \{(c,k) *^{\sigma} (b,-1) : a < c,\ k \in \{0,-1\}\} \\ &= \bigwedge \{(c * b,-1)\,,: a < c\} \\ &= (a * b, 1)\,.\end{aligned}$$

□

Proposition 12. *Let $\boldsymbol{P}$ be the interval product algebra. Then implications $\rightarrow^{\sigma}$ and $\rightarrow^{\pi}$ are given by the following tables:*

$$(a,i) \rightarrow^{\sigma} (b,j) = \begin{cases} (a \rightarrow b, 0) & \text{if } (i,j) = (0.0), \\ (1,0) & \text{if } a \leq b \text{ and } i < j, & (a) \\ \left(\frac{b}{a}, 1\right) & \text{If } b < a \text{ and } i < j, & (b) \\ (1,0) & \text{If } a < b \text{ and } i \geq j, & (c) \\ \left(\frac{b}{a}, -1\right) & \text{If } b \leq a,\ i \geq j \text{ and } (i,j) \neq (0.0). & (d) \end{cases}$$

and

$$(a,i) \rightarrow^{\pi} (b,j) = \begin{cases} (1,0) & \text{if } a \leq b \text{ and } i = j = -1 & (e) \\ \left(\frac{b}{a}, 1\right) & \text{if } b < a \text{ and } i = j = -1 & (f) \\ (1,0) & \text{if } a \leq b \text{ and } i = j = 1 & (g) \\ \left(\frac{b}{a}, 1\right) & \text{if } b < a \text{ and } i = j = 1 & (h) \\ (a,i) \rightarrow^{\sigma} (b,j) & \text{otherwise.} \end{cases}$$

Proof. Now we calculate $\rightarrow^{\sigma}$.

(a) If $a \leq b$, suppose that $(i,j) = (0,1)$, then

$$((a,i)\,,(b,j)) \in K\left(\left([0,1]^{d} \times [0,1]\right)^{\sigma}\right).$$

Therefore,

$$\begin{aligned}(a,i) \rightarrow^{\sigma} (b,j) &= \bigwedge \{(c \rightarrow d, 0) : c \leq a, b < d\} \\ &= \bigwedge \{(1,0) : c \leq a, b < d\} \\ &= (1,0)\,.\end{aligned}$$

If $(i,j) \in \{(-1,0)\,,(-1,1)\}$ the proof is similar.

(b) If $b < a$, suppose that $(i,j) = (0,1)$. Therefore,

$$\begin{aligned}(a,0) \rightarrow^{\sigma} (b,1) &= \bigwedge \{(c \rightarrow d, 0) : c \leq a, b < d\} \\ &= \bigwedge \left\{\left(\tfrac{d}{c}, 0\right) : b < d \leq c \leq a\right\} \\ &= \left(\tfrac{b}{a}, 1\right).\end{aligned}$$

If $(i,j) \in \{(-1,0),(-1,1)\}$ the proof is similar.

(c) If $a < b$ and $(i,j) = (-1,-1)$,

$$\begin{aligned}(a,-1) \to^{\sigma} (b,-1) &= \bigvee \{(c,k) \to^{\sigma} (d,l) : a \leq c, d < b, \\ &\quad k \in \{0,-1\}, l \in \{0,1\}\} \\ &= \bigvee \{(a,-1) \to^{\sigma} (d,l) : d < b,\ l \in \{0,1\}\} \\ &= \bigvee \{(1,0) : d < b,\ l \in \{0,1\}\} \\ &= (1,0).\end{aligned}$$

If $(i,j) \in \{(0,-1),(1,0),(1,1),(1,-1)\}$ the proof is similar.

(d) If $b \leq a$ and $(i,j) = (-1,-1)$,

$$\begin{aligned}(a,-1) \to^{\sigma} (b,-1) &= \bigvee \{(a,-1) \to^{\sigma} (d,l) : d < b,\ l \in \{0,1\}\} \\ &= \bigvee \left\{\left(\tfrac{d}{a},1\right) : d < b\right\} \\ &= \left(\tfrac{b}{a},-1\right).\end{aligned}$$

If $(i,j) \in (i,j) \in \{(0,-1),(1,0),(1,1),(1,-1)\}$ the proof is similar.

Since $\to^{\sigma}$ and $\to^{\pi}$ coincide in

$$K\left(\left([0,1]^d \times [0,1]\right)^{\sigma}\right) \cup O\left(\left([0,1]^d \times [0,1]\right)^{\sigma}\right)$$

we only have to prove (e)(f)(g)(h).

(e) Since $a \leq b$,

$$\begin{aligned}(a,-1) \to^{\pi} (b,-1) &= \bigwedge \{(c,k) \to^{\pi} (b,-1) : c < a \leq b,\ k \in \{0,1\}\} \\ &= (1,0).\end{aligned}$$

(f) Since $b < a$,

$$\begin{aligned}(a,-1) \to^{\pi} (b,-1) &= \bigwedge \{(c,k) \to^{\pi} (b,-1) : c < a,\ k \in \{0,1\}\} \\ &= \bigwedge \{(c,k) \to^{\sigma} (b,-1) : c < a,\ k \in \{0,1\}\} \\ &= \bigwedge \left\{\left(\tfrac{b}{c},-1\right) : b \leq c < a\right\} \cup \{(1,0)\} \\ &= \left(\tfrac{b}{a},1\right).\end{aligned}$$

(g) and (h) follow in a similar way. □

Corollary 13. *The product and the implications of* $\boldsymbol{P}$ *are not smooth operators.*

Theorem 14. *Neither $\boldsymbol{P}^\sigma$ nor $\boldsymbol{P}^\pi$ are* **BL**-*algebras.*

Proof. In neither case the equation $a\odot(a \to b) = b\odot(b \to a)$ is satisfied. For $\boldsymbol{P}^\sigma$ consider

$$\begin{aligned}\left(\tfrac{1}{2},0\right) *^\sigma \left(\left(\tfrac{1}{2},0\right) \to^\sigma \left(\tfrac{1}{2},1\right)\right) &= \left(\tfrac{1}{2},1\right) *^\sigma (1,0)\\ &= \left(\tfrac{1}{2},1\right),\end{aligned}$$

and

$$\begin{aligned}\left(\tfrac{1}{2},1\right) *^\sigma \left(\left(\tfrac{1}{2},1\right) \to^\sigma \left(\tfrac{1}{2},0\right)\right) &= \left(\tfrac{1}{2},1\right) *^\sigma (1,-1)\\ &= \left(\tfrac{1}{2},-1\right).\end{aligned}$$

For the case of $\boldsymbol{P}^\pi$ consider

$$\begin{aligned}\left(\tfrac{1}{2},-1\right) *^\pi \left(\left(\tfrac{1}{2},-1\right) \to^\pi \left(\tfrac{1}{2},1\right)\right) &= \left(\tfrac{1}{2},-1\right) *^\pi (1,0)\\ &= \left(\tfrac{1}{2},-1\right)\end{aligned}$$

and

$$\begin{aligned}\left(\tfrac{1}{2},1\right) *^\pi \left(\left(\tfrac{1}{2},1\right) \to^\pi \left(\tfrac{1}{2},-1\right)\right) &= \left(\tfrac{1}{2},1\right) *^\pi (1,-1)\\ &= \left(\tfrac{1}{2},1\right).\end{aligned}$$

□

Summarizing the results of this section, we get the following proposition (see Fig. 3).

Proposition 15. *Let $\mathcal{V}$ be a subvariety of* **ŁΠG**. *Then the following statements are valid:*

1. *$\mathcal{V}$ is σ-canonical and π-canonical if and only if it is generated by a finite set of finite algebras.*
2. *If $\mathcal{V}$ is not generated by a finite set of finite algebras, then $\mathcal{V}$ is not σ-canonical*
3. *If $\mathcal{V}$ is not generated by a finite set of finite algebras, then:*
   (a) *If $\mathcal{V}$ has a non trivial product* **BL**-*algebra then $\mathcal{V}$ is not π-canonical.*
   (b) *If there is some $n \geq 1$, such that $S_n^\omega \in \mathcal{V}$ then $\mathcal{V}$ is not π-canonical.*

Acknowledgments. I would like to thank Manuela Busaniche for her valuable comments made during the making of this paper. They have helped to improve this work. I also want to thank the anonymous referee for his comments and observations on the structure of this paper.

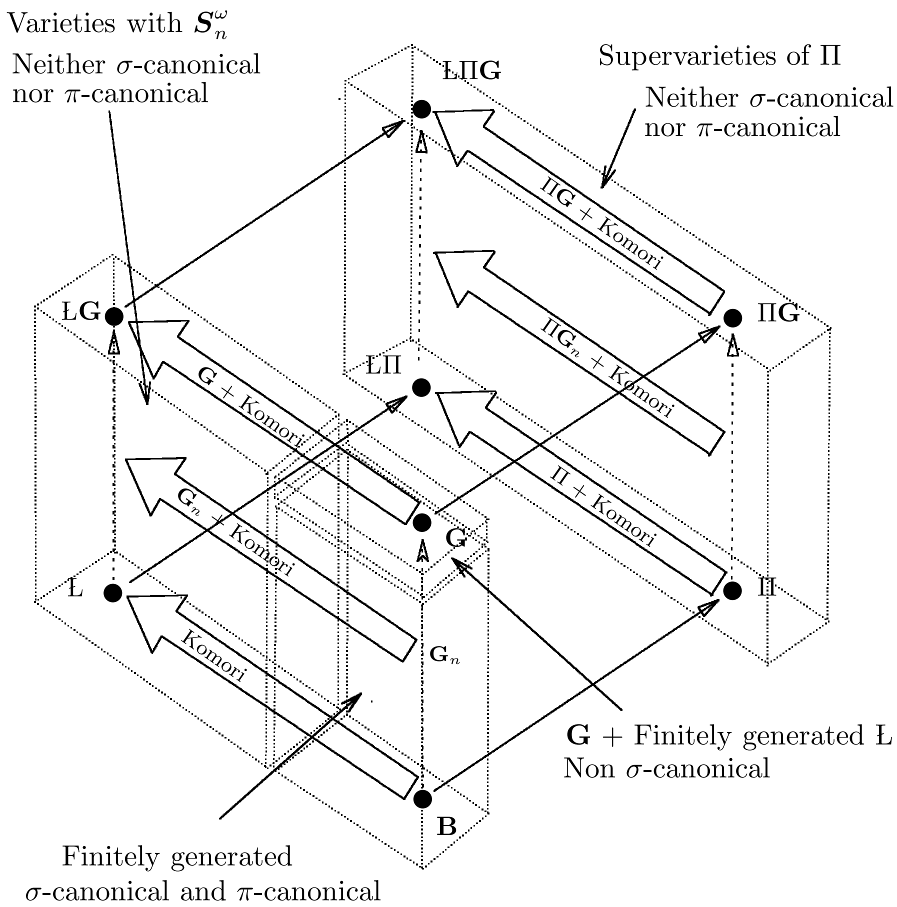

Fig. 3

References

[1] R. L. Cignoli, A. Torrens, *An algebraic analysis of product logic*, Multiple valued Logic **5** (2000), pp. 45–65.

[2] A. Di Nola, F. Esteva, P. Garcia, L. Godo, S. Sessa, *Subvarieties of BL-algebras generated by single-component chains*, Archive for Mathematical Logic **41**, 7 (2002), pp. 673–685.

[3] F. Esteva, L. Godo, *Monoidal t-norm based logic: towards a logic for left-continuous*

t-norms, Fuzzy Sets and Systems **124**, 3 (2001), pp. 271–288.

[4] M. Gehrke, J. Harding, *Bounded lattice expansions*, Journal of Algebra **238**, 1 (2001), pp. 345–371.

[5] M. Gehrke, B. Jónsson, *Bounded distributive lattices with operators*, Mathematica Japonica **40**, 2 (1994), pp. 207–215.

[6] M. Gehrke, B. Jónsson, *Bounded distributive lattice expansions*, Mathematica Scandinavica **94**, 1 (2004), pp. 13–45.

[7] M. Gehrke, H. Priestley, *Canonical extensions of double quasioperator algebras: An algebraic perspective on duality for certain algebras with binary operations*, Journal of Pure and Applied Algebra **209**, 1 (2007), pp. 269–290.

[8] M. Gehrke, H. Priestley, *Non-canonicity of MV-algebras*, Houston Journal of Mathematics **28**, 3 (2002), pp. 449–455.

[9] P. Hájek, *Basic fuzzy logic and BL-algebras*, Soft Computing **2**, 3 (1998), pp. 124–128.

[10] P, Hájek, Metamathematics of Fuzzy Logic, Trends in Logic - Studia Logica Library **4**, Kluwer Academic Publishers (1998).

[11] P. Jipsen, C. Tsinakis, *A Survey of Residuated Lattices*, Ordered Algebraic Structures (2002), Kluwer Academic Publishers, pp. 19-56.

[12] B. Jónsson, A. Tarski, *Boolean algebras with operators*, American Journal of Mathematics **73** (1951), pp. 891–939.

[13] B. Jónsson, A. Tarski, *Boolean algebras with operators II*, American Journal of Mathematics **74** (1952), pp. 127–162.

[14] Y. Komori, *Super-Lukasiewics propositional logics*, Nagoya Mathematical Journal **84** (1981), pp. 119–133.

[15] H. Ono, *Substructural logics and residuated lattices - an introduction*, Trends in Logic **21** (2003), 50 Years of Studia Logica. Kluwer Academic Publishers, pp. 193–228.

[16] E. Turunen, Mathematics Behind Fuzzy Logic, Advances in soft Computing, Physica-Verlag, Heidelberg (1999).

CONICET and
Departamento de Matemáticas.
Facultad de Ciencias Exactas
Universidad Nacional del Centro
Pinto 399
7000 Tandil. Argentina
cabrer@exa.unicen.edu.ar