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SUBMINIMAL LOGIC AND WEAK ALGEBRAS

A b s t r a c t. In this paper we investigate the implication-less fragment of Johansson's minimal logic. We call it subminimal logic and we study its associated algebras, which we call weak algebras. We prove the algebraic Glivenko theorem, soundness and completeness for this logic.

1. Introduction

In 1929 Glivenko showed that if there is a classical derivation of a propositional formula, then there is an intuitionistic derivation of its double negation (see [3]). There is an algebraic version of Glivenko's theorem stating that (i) the set of regular elements of a Heyting algebra A admits a Boolean algebra structure, denoted by Reg(A), and (ii) the double negation defines an homomorphism from A onto Reg(A).

Extending Glivenko's theorem to Johansson's minimal logic (that is, the logic that arises from intuitionistic logic when the EASQ rule is omitted)

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does not work because the proof fails in the case of the conditional (the conditional $(\neg\neg\varphi \rightarrow \neg\neg\psi) \rightarrow \neg\neg(\varphi \rightarrow \psi)$ is not derivable in minimal logic).

In this paper we investigate the implication-less fragment of Johansson's minimal logic. We call it subminimal logic and its associated algebras will be called weak algebras. Then we prove soundness and completeness and the algebraic Glivenko theorem for this logic.

2. Intuitionistic and Minimal Logic

The following version of intuitionistic logic $\mathcal{I}$ was introduced by Gentzen in 1934 (see [6], [8]).

The language of $\mathcal{I}$ is builded recursively from the set Var of propositional variables by means of the connectives $\wedge$, $\vee$, $\rightarrow$ and $\neg$.

A *sequent* of $\mathcal{I}$ is an expression of the form $\alpha_1,...,\alpha_n \vdash \beta$, where $\alpha_1,...,\alpha_n$ and β are formulas. In the following, Greek capital letters denote finite, possibly empty sequences of formulas. The sequents of the form $\alpha \vdash \alpha$ are called *initial* sequents.

The following are the *rules* of $\mathcal{I}$.

Weakening rules:

$$\text{(W1)} \quad \frac{\Gamma, \Sigma \vdash \varphi}{\Gamma, \alpha, \Sigma \vdash \varphi} \qquad \text{(W2)} \quad \frac{\Gamma \vdash}{\Gamma \vdash \alpha}$$

Contraction rule:

$$\text{(C)} \quad \frac{\Gamma, \alpha, \alpha, \Sigma \vdash \varphi}{\Gamma, \alpha, \Sigma \vdash \varphi}$$

Exchange rule:

$$\text{(E)} \quad \frac{\Gamma, \alpha, \beta, \Sigma \vdash \varphi}{\Gamma, \beta, \alpha, \Sigma \vdash \varphi}$$

Cut rule:

$$\text{(Cut)} \quad \frac{\Gamma \vdash \varphi \qquad \Sigma, \varphi, \Pi \vdash \psi}{\Sigma, \Gamma, \Pi \vdash \psi}$$

Conjunction rules:

$$\text{(CE1)} \quad \frac{\Gamma, \alpha, \Sigma \vdash \varphi}{\Gamma, \alpha \wedge \beta, \Sigma \vdash \varphi} \qquad \text{(CE2)} \quad \frac{\Gamma, \alpha, \Sigma \vdash \varphi}{\Gamma, \beta \wedge \alpha, \Sigma \vdash \varphi}$$

$$\text{(CI)} \quad \frac{\Gamma \vdash \varphi \qquad \Gamma \vdash \psi}{\Gamma \vdash \varphi \wedge \psi}$$

Disjunction rules:

$$\text{(DE)} \quad \frac{\Gamma, \alpha, \Sigma \vdash \varphi \qquad \Gamma, \beta, \Sigma \vdash \varphi}{\Gamma, \alpha \vee \beta, \Sigma \vdash \varphi}$$

$$\text{(DI1)} \quad \frac{\Gamma \vdash \alpha}{\Gamma \vdash \alpha \vee \beta} \qquad \text{(DI2)} \quad \frac{\Gamma \vdash \alpha}{\Gamma \vdash \beta \vee \alpha}$$

Conditional rules:

$$\text{(IE)} \quad \frac{\Gamma \vdash \alpha \qquad \Pi, \beta, \Sigma \vdash \gamma}{\Pi, \alpha \to \beta, \Gamma, \Sigma \vdash \gamma} \qquad \text{(II)} \quad \frac{\Gamma, \alpha \vdash \beta}{\Gamma \vdash \alpha \to \beta}$$

Negation rules:

$$\text{(NE)} \quad \frac{\Gamma \vdash \alpha}{\neg\alpha, \Gamma \vdash} \qquad \text{(NI)} \quad \frac{\Gamma, \alpha \vdash}{\Gamma \vdash \neg\alpha}.$$

We define the notion of derivation as usual: a derivation of the sequent $\Gamma \vdash \beta$ is a finite number of applications of rules to the formulas of Γ with last formula β.

We obtain minimal logic $\mathcal{M}$ by dropping (W2).

Lemma 1. *In $\mathcal{M}$ we obtain the following derived rule:*

$$\textit{(NSI)} \quad \frac{\Gamma, \varphi \vdash \psi \qquad \Gamma, \varphi \vdash \neg\psi}{\Gamma \vdash \neg\varphi}.$$

Proof. Suppose $\Gamma, \varphi \vdash \psi$ and $\Gamma, \varphi \vdash \neg\psi$. By (CI), $\Gamma, \varphi \vdash \psi \wedge \neg\psi$ and by (NE): () $\neg(\psi \wedge \neg\psi), \Gamma, \varphi \vdash$. Also, from the initial sequent $\psi \vdash \psi$ and by the rules (NE), (CE1), (E), (CE2),(C) and (NI) we obtain: (‡) $\vdash \neg(\psi \wedge \neg\psi)$ (see [6]). By the cut rule, from () and (‡) we prove $\Gamma, \varphi \vdash$ from where, by (NI) we have $\Gamma \vdash \neg\varphi$. □

3. Subminimal Logic

If we drop the conditional from $\mathcal{M}$, then the logic obtained will allow us to prove Glivenko theorem.

The subminimal logic $\mathcal{S}$ is builded with the same rules as $\mathcal{M}$, except the rules for the conditional. We will use (NSI) instead of (NI) and (NE). Note that (NSI) is available in $\mathcal{S}$ because we have not used the conditional rules in our previous derivation of (NSI). The set of formulas of $\mathcal{S}$ will be denoted by $\mathcal{F}_{\mathcal{S}}$.

Lemma 2. *1) Modus Ponendo Tollens is derivable in $\mathcal{S}$:*

$$\text{(MPT)} \quad \frac{\Gamma \vdash \neg(\alpha \wedge \beta) \qquad \Gamma \vdash \alpha}{\Gamma \vdash \neg\beta}$$

2) A sequent $\alpha_1, \alpha_2, ..., \alpha_n \vdash \beta$ is provable in $\mathcal{S}$ if and only if the sequent $\alpha_1 \wedge \alpha_2 \wedge ... \wedge \alpha_n \vdash \beta$ is provable in $\mathcal{S}$ (see [8], Proposition 2.1).

Proof. 1. From sequents $\alpha \vdash \alpha$, $\beta \vdash \beta$ and $\neg(\alpha \wedge \beta) \vdash \neg(\alpha \wedge \beta)$ we obtain (MPT) by means of rules (W1), (CI) and (NSI). □

Lemma 3. *For every α, β and γ in $\mathcal{F}_{\mathcal{S}}$, the following properties hold:*

(i) $\alpha \vdash \neg(\beta \wedge \neg\beta)$;

(ii) $\neg(\alpha \wedge \neg\alpha) \vdash \neg(\gamma \wedge \neg\gamma)$;

(iii) $\alpha \wedge \neg(\alpha \wedge \beta) \vdash \neg\beta$;

(iv) $\alpha \vdash \neg\neg\alpha$;

(v) $\neg(\alpha \vee \beta) \vdash \neg\alpha$;

(vi) $(\neg\alpha \wedge \neg\beta) \wedge (\alpha \vee \beta) \vdash \neg\neg(\gamma \wedge \neg\gamma)$.

Proof. (i) Apply (NSI) to the sequents

$$\alpha,\ \beta \wedge \neg\beta \vdash \beta \quad \text{and} \quad \alpha,\ \beta \wedge \neg\beta \vdash \neg\beta.$$

(ii) Substitute $\neg(\alpha \wedge \neg\alpha)$ for α and γ for β in (i).

(iii) It follows from (MPT).

(iv) From (i) we have: $\alpha \vdash \neg(\alpha \wedge \neg\alpha)$. Then, $\alpha \vdash \alpha \wedge \neg(\alpha \wedge \neg\alpha)$ and we deduce from (iii) that $\alpha \wedge \neg(\alpha \wedge \neg\alpha) \vdash \neg\neg\alpha$.

(v) Using rule (DI1) it follows that: $\alpha \vdash \alpha \vee \beta$. Then, $\alpha, \neg(\alpha \vee \beta) \vdash \alpha \vee \beta$ and $\alpha, \neg(\alpha \vee \beta) \vdash \neg(\alpha \vee \beta)$. The result follows from (NSI).

(vi) Using rules (CE1) and (CE2) we have that

$$\alpha \wedge \neg\alpha \wedge \neg\beta \vdash \alpha \wedge \neg\alpha \quad \text{and} \quad \beta \wedge \neg\alpha \wedge \neg\beta \vdash \beta \wedge \neg\beta.$$

By (iv) we can find also derivations of $\neg\neg(\alpha \wedge \neg\alpha)$ and $\neg\neg(\beta \wedge \neg\beta)$ from $\alpha \wedge \neg\alpha \wedge \neg\beta$ and from $\beta \wedge \neg\alpha \wedge \neg\beta$ respectively.

We observe, by (ii), that $\neg\neg(\alpha \wedge \neg\alpha)$ and $\neg\neg(\beta \wedge \neg\beta)$ are equivalent, and equivalent to, say, $\neg\neg(\gamma \wedge \neg\gamma)$. Then, by (DE), we have:

$(\alpha \vee \beta) \wedge \neg\alpha \wedge \neg\beta \vdash \neg\neg(\gamma \wedge \neg\gamma)$. □

4. Weak algebras

It is well known that the Lindenbaum algebra of the intuitionistic calculus $\mathcal{I}$ is a *Heyting algebra*.

A *generalized Heyting algebra* (*gH-algebra* for short) (see [7]) is a lattice $\langle A, \wedge, \vee \rangle$ such that for every pair a, b of elements of A there exists the maximum of the set $\{z \in A : a \wedge z \leq b\}$, denoted by $a \to b$. A *Heyting algebra* is a gH–algebra with a minimum 0. If we require the weaker condition of the existence of $a \to b$ only for the case $b = 0$, then we call A a *pseudocomplemented* algebra (see [1]).

The algebraic counterpart of minimal logic is the structure of *gH*-algebra. If A is a *gH*-algebra, then we can define in A a negation such that the following condition holds (see [2] section 2, Lemma 1):

$$(x \to y) \to ((x \to \neg y) \to \neg x) = 1.$$

In fact, choose an element $t \in A$ and define, for each $x \in A$,

$$\neg x = x \to t.$$

The algebraic semantics for the system $\mathcal{S}$ is given by the variety of weak algebras, that we study in this section.

Definition 1. The system $\langle A, \wedge, \vee, \neg \rangle$ is a weak algebra if the following conditions hold:

(1) $\langle A, \wedge, \vee \rangle$ is a distributive lattice;

(2) $\neg(x \wedge y) \wedge x \leq \neg y$;

(3) $\neg(x \vee y) = \neg x \wedge \neg y$;

(4) $x \leq \neg\neg x$.

We prove some technical results.

Lemma 4. *The following properties are true in weak algebras:*

For every x, y in A

(a) $x \leq y$ *implies* $\neg y \leq \neg x$;

(b) $\neg\neg\neg x = \neg x$;

(c) $y \leq \neg(x \wedge \neg x)$;

(d) $\neg(x \wedge \neg x) = \neg(y \wedge \neg y)$.

Proof. (a) From (3).

(b) From (4): $\neg x \leq \neg\neg(\neg x)$. From (4) and (a), $\neg\neg\neg x \leq \neg x$.

(c) From (2) and (a) $\neg x \wedge x \leq \neg(x \wedge y) \wedge x \leq \neg y$. Again by (a) and using (4): $\neg(x \wedge \neg x) \geq \neg\neg y \geq y$.

(d) Put $\neg(y \wedge \neg y)$ instead of y in (c). □

Remark 1. Properties (c) and (d) above allows us to call 1 the element $\neg(x \wedge \neg x)$.

Lemma 5. *For every x, y in A*

(i) $x \leq \neg y$ *if and only if there exists* z *such that* $x \wedge y \leq z \wedge \neg z$ *if and only if* $\neg(x \wedge y) = 1$;

(ii) $\neg(x \wedge y) = 1$ *if and only if* $\neg(\neg\neg x \wedge y) = 1$;

(iii) $\neg\neg(x \wedge y) = \neg\neg x \wedge \neg\neg y$.

Proof. (i) Suppose $x \leq \neg y$. Then, $x \wedge y \leq \neg y \wedge y$ and, by Lemma 4 (a) and Remark 1: $\neg(x \wedge y) \geq 1$, that is, $\neg(x \wedge y) = 1$. This last equality implies $x \leq \neg y$, by (2) in Definition 1.

(ii) If $\neg(x \wedge y) = 1$ then, by (i), $y \leq \neg x$ and so $\neg y \geq \neg\neg x$. Again by (i), $\neg(\neg\neg x \wedge y) = 1$.

The converse follows from (4) in Definition 1 and anti-monotonicity of $\neg$.

(iii) By monotonicity of $\neg\neg$, $\neg\neg(x \wedge y) \leq \neg\neg x \wedge \neg\neg y$.

To prove the other inequality, we can consider the equality: $\neg((x \wedge y) \wedge \neg(x \wedge y)) = 1$. Then, by (ii), $\neg(\neg\neg x \wedge (y \wedge \neg(x \wedge y))) = 1$. Again by (ii), $\neg(\neg\neg x \wedge (\neg\neg y \wedge \neg(x \wedge y))) = \neg((\neg\neg x \wedge \neg\neg y) \wedge \neg(x \wedge y)) = 1$ and so, by (i), $\neg\neg x \wedge \neg\neg y \leq \neg\neg(x \wedge y)$. □

We remark that even in the case of the weak algebra A having a least element 0, the negation may not be a pseudocomplement on A (see [1]), as the following example proves. Consider $\mathbf{A} = \langle A, \wedge, \vee, \neg \rangle$ such that $A = \{0, a, b, c, 1\}$, where $c = a \wedge b$, $1 = a \vee b$ and the negation is given by: $\neg a = b$, $\neg b = a$, $\neg c = \neg 0 = 1$, $\neg 1 = c$.

Also, every distributive lattice with the greatest element 1 is a weak algebra, by putting $\neg x = 1$, for every x.

Moreover, a weak algebra can be thought as a "generalized" pseudocomplemented algebra as the following Lemma states.

Lemma 6. *Let $\langle A, \wedge, \vee \rangle$ be a distributive lattice and $t \in A$. Suppose that for every $x \in A$ there exists $\neg_t x$, the maximum of the set $\{y : \ y \wedge x \leq t\}$. Then $\langle A, \wedge, \vee, \neg_t \rangle$ is a weak algebra.*

Proof. First we prove (2), that is, $\neg_t(x \wedge y) \wedge x \leq \neg_t y$. It suffices to show that $(\neg_t(x \wedge y) \wedge x) \wedge y = \neg_t(x \wedge y) \wedge (x \wedge y) \leq t$, which is obvious.

It is easy to see that $u \leq v$ implies $\neg_t v \leq \neg_t u$ (anti-monotonicity of $\neg_t$). We use this property to prove the inequality $\neg_t(x \vee y) \leq \neg_t x \wedge \neg_t y$. To prove the other inequality of (3) we show that $(\neg_t x \wedge \neg_t y) \wedge (x \vee y) \leq t$. In fact, $(\neg_t x \wedge \neg_t y) \wedge (x \vee y) = (\neg_t x \wedge x \wedge \neg_t y) \vee (\neg_t x \wedge \neg_t y \wedge y) \leq (\neg_t x \wedge x) \vee (\neg_t y \wedge y) \leq t \vee t = t$.

From the fact that $\neg_t x \wedge x \leq t$ we deduce the property $x \leq \neg_t \neg_t x$. □

Corollary 7. *Let $\langle A, \wedge, \vee, \rightarrow \rangle$ be a gH-algebra. Then, for every $t \in A$ $\langle A, \wedge, \vee, \neg_t \rangle$ is a weak algebra, where $\neg_t x = x \rightarrow t$.*

Conversely, in every weak algebra we can choose an element t such that for every x there exists $x \rightarrow t$.

Lemma 8. *Let $\langle A, \wedge, \vee, \neg \rangle$ be a weak algebra, let $t = \neg 1$. Then for every $x \in A$ the maximum of the set $\{y : \ y \wedge x \leq t\}$ is $\neg x$.*

Proof. First, we have $\neg x \wedge x \leq \neg x \wedge \neg\neg x = \neg 1$. Second, if $y \wedge x \leq \neg 1$ then $\neg(y \wedge x) = 1$ and so, from (2) in Definition 1, $y \leq \neg x$. □

5. Congruences in weak algebras

In which follows we state the relationship between filters and congruences in weak algebras and establish some conditions relative to subdirectly irreducible weak algebras.

We recall that a *filter* in a lattice is an increasing non empty set closed by infimum. In a dual manner, an *ideal* is a decreasing non empty set closed by supremum. We denote $[x)$ (respectively $(x]$) the filter of the elements greater or equal than x (respectively the ideal of the elements least or equal than x).

The lattice congruence R_F associated to a filter F is defined by:
$x\, R_F\, y$ if and only if there exists an element $u \in F$ such that $x \wedge u = y \wedge u$.

The lattice congruence R_I associated to an ideal I is defined by: $x\, R_I\, y$ if and only if there exists an element $v \in I$ such that $x \vee v = y \vee v$.

Lemma 9. *Let $\langle A, \wedge, \vee, \neg\rangle$ be a weak algebra, F a filter of A. Then, the relation R_F is a congruence of weak algebras.*

Proof. We only have to prove that R_F preserves negation.

Let $x\, R_F\, y$ and let u be such that $x \wedge u = y \wedge u$. Then, $\neg y \wedge x \wedge u = \neg y \wedge y \wedge u \leq \neg y \wedge y$, from where $\neg(\neg y \wedge x \wedge u) = 1$. By (i) Lemma 5 it follows that $\neg y \wedge u \leq \neg x$, and so $\neg y \wedge u \leq \neg x \wedge u$. The other inequality is analogous. □

Nevertheless, not every congruence is of the form R_F for some F, as it can be proven by the following

Example 1. Let $\mathbf{L} = \langle L, \wedge, \vee, \neg\rangle$ be such that $L = \{0, x, y, z, 1\}$, where $0 = x \wedge y$, $z = x \vee y$ and the negation is given by: $\neg x = \neg z = \neg 1 = y$, $\neg y = \neg 0 = 1$. If R is the congruence given by $\{\{x, z\}, \{0, y\}, \{1\}\}$, then $R \neq R_F$ for every filter F.

The same lattice congruence can be used as an example of one that corresponds to an ideal but does not preserve negation, provided that we change on L the weak algebra structure, that is, the negation.

Example 2. Define on $L = \{0, x, y, z, 1\}$ the negation given by the pseudocomplement, that is: $\neg x = y$, $\neg y = x$, $\neg 1 = \neg z = 0$, $\neg 0 = 1$. Then, $R = R_I$ for $I = [y)$ and we have that $y\, R\, 0$ but $\neg y$ is not related by R with $1 = \neg 0$.

Recall the well known property in Universal Algebra (see [1], Th.3, I, 9) that an algebra is subdirectly irreducible if and only if there is a minimal non trivial congruence.

Remark 2. It is easy to see that if a weak algebra A is such that $|A| \leq 2$, then A is simple, thus subdirectly irreducible. So, it suffices to deal with algebras under the condition $|A| > 2$.

Following the "only if" part of the characterization theorem for subdirectly irreducibles in pseudocomplemented algebras (see [1], Th. 1, VIII, 5), we can prove the following

Theorem 10. *Let L be a subdirectly irreducible weak algebra and $|L| > 2$. Then, there exists $p \in L$ such that*

1. *1 covers p and p is the maximum of the elements less than 1;*
2. *For every x, if $x \leq p$ and $x \not\leq \neg p$, then $x \vee \neg x = p$;*
3. *For every congruence R, there is some $r \in L$ such that $\neg\neg r = r$ and $r\, R\, p$;*
4. *$\neg\neg p = 1$*

Proof. Let R_0 be the minimal non trivial congruence. Let p be such that $(p, 1) \in R_0$. Then, 1 covers p. Moreover, 1 is a prime element, because if $x \vee y = 1$, then the congruences R_{x^+}, R_{y^+} associated to the filters $[x)$ and $[y)$ must contain R_0, from where $(p, 1) \in R_{x^+} \cap R_{y^+}$, which implies $p = 1$, absurd. So, 1) is proved.

To prove 2), let x be such that $x \leq p$ and $x \not\leq \neg p$ and suppose $x \vee \neg x = s < p$. Then, $\neg s = \neg x \wedge \neg\neg x = \neg 1$ and so $\neg\neg s = \neg\neg 1 = 1$. Consider the lattice congruences R_{p^-} and R_{s^+} associated to, respectively, the ideal $(p]$ and the filter $[s)$. Put $R_1 = R_{p^-} \cap R_{s^+}$. Then, R_1 is not trivial because $(s, p) \in R_1$. We will show that R_1 is a weak algebra congruence, for which it is enough to show that R_{p^-} preserves negation.

Let u, v be such that $(u, v) \in R_1$. We have $\{\neg u \vee p, \neg v \vee p\} \subseteq \{p, 1\}$. Suppose $\neg u \vee p = 1$. Then (as 1 is prime) $\neg u = 1$. We have $\neg u R_{s^+} \neg v$ (because R_{s^+} preserves negation), that is, $1 R_{s^+} \neg v$, from which $s \leq \neg v$ follows. Then $\neg\neg s = 1 \leq \neg v$ which implies $\neg v \vee p = 1 = \neg u \vee p$, that is, $(\neg u, \neg v) \in R_1$.

In that case $(p, 1) \notin R_1$, because $p \vee p \neq 1 \vee p$, from where $R_0 \nsubseteq R_1$, absurd.

The condition 3) is obviously true by putting $r = 1$.

In order to prove 4), we need the equality (*) $\neg\neg p \wedge \neg p = \neg 1$, that holds by (iii), Lemma 5. If $\neg p = p$ then $p = 1$, absurd. If $\neg p = 1$, then $\neg u = 1$ for every $u \leq p$. So, there is a congruence that identifies all the elements less or equal to p and that does not contains R_0, absurd. Thus, we see that $\neg p < p$, which implies $\neg\neg p \geq \neg p$. Then we deduce from (*) that $\neg\neg p = 1$. □

Remark 3. The conditions 1) and 2) of the previous theorem are not sufficient, as it is shown by example 1 above. Indeed, conditions 1 and 2 hold but R_{z^+} is not a minimal non trivial congruence because $R_{z^+} \subsetneq R$. So, there is no non trivial minimal congruence.

Theorem 11. *Let L be a weak algebra and $|L| > 2$. Then, L is subdirectly irreducible if and only if 1), 3) and 4) above hold for some p.*

Proof. Let be r such that $r\,R\,p$. Then, $\neg\neg r = r\,R\,\neg\neg p = 1$ which implies $p\,R\,1$, that is, $R \supseteq R_0$, where R_0 is the congruence that only identifies p and 1. □

6. Algebraic Glivenko theorem

In this section we prove Glivenko theorem for the variety of weak algebras.

Theorem 12. *(Glivenko) Let $Reg(A)$ be the set of regular elements defined by $Reg(A) = \{x \in A,\ x = \neg\neg x\}$. Then, the system*

$$\langle Reg(A), \wedge, \vee_r, \neg, \neg 1, 1\rangle$$

is a Boolean algebra, where $\vee_r$ is defined by $x \vee_r y = \neg\neg(x \vee y)$. Moreover, the application $\neg\neg : A \longrightarrow Reg(A)$ is an homomorphism of weak algebras.

Proof. It is easy to see that $\wedge$ is closed in $Reg(A)$, because an element t is regular if and only if it is of the form $t = \neg z$ for some z. We prove that $\vee_r$ is a supremum in $Reg(A)$. In fact,if $\neg x$, $\neg y$ are regular elements of A, then $\neg x \vee_r \neg y \geq \neg x \vee \neg y$, so, $\neg x \vee_r \neg y$ is an upper bound for $\{\neg x, \neg y\}$.

Also, if $\neg z$ is a common upper bound of $\neg x$, $\neg y$, then $\neg x \vee \neg y \leq \neg z$, so $\neg\neg(\neg x \vee \neg y) \leq \neg\neg\neg z = \neg z$.

It is easy to see that, for a regular element $\neg x$, $\neg x \vee_r \neg\neg x = 1$ and $\neg x \wedge \neg\neg x = \neg 1$.

To prove that $\neg\neg$ preserves supremum we use (3) of Definition 1 and (b) of Lemma 4. The property (iii) of Lemma 5 states that $\neg\neg$ preserves infimum. Trivially $\neg\neg(\neg x) = \neg(\neg\neg x)$. So, $\neg\neg$ is an homomorphism. □

7. Lindenbaum algebra of $\mathcal{S}$

Consider the relation $\vdash$ in the set $\mathcal{F}_{\mathcal{S}}$. It is obviously reflexive and, by the cut rule, it is also transitive. That is, $\vdash$ is a preorder relation. Then, the relation $\dashv\vdash$ given by: $\alpha \dashv\vdash \beta$ if and only if $\alpha \vdash \beta$ and $\beta \vdash \alpha$ is an equivalence relation and $\vdash$ induces an order $\preceq$ in the quotient $\mathbb{W} = \mathcal{F}_{\mathcal{S}}/\dashv\vdash$. That is, for $\overline{\alpha}$, $\overline{\beta}$ in $\mathbb{W}$, $\overline{\alpha} \preceq \overline{\beta}$ if and only if $\alpha \vdash \beta$. Moreover, we can prove the properties of $\dashv\vdash$ stated in the next Lemma.

Lemma 13. *Let $\alpha, \beta, \varphi, \psi$ be formulas of $\mathcal{S}$ and suppose $\alpha \dashv\vdash \beta$ and $\varphi \dashv\vdash \psi$. Then, $\alpha \wedge \varphi \dashv\vdash \beta \wedge \psi$, $\alpha \vee \varphi \dashv\vdash \beta \vee \psi$ and $\neg\alpha \dashv\vdash \neg\beta$. So, the relation $\dashv\vdash$ is a congruence on the absolutely free algebra $\langle \mathcal{F}_{\mathcal{S}}, \wedge, \vee, \neg \rangle$.*

Theorem 14. *The system $\langle \mathbb{W}, \wedge, \vee, \neg \rangle$ is a weak algebra, where the operations are defined by: $\overline{\alpha} \wedge \overline{\beta} = \overline{\alpha \wedge \beta}$, $\overline{\alpha} \vee \overline{\beta} = \overline{\alpha \vee \beta}$, $\neg\overline{\alpha} = \overline{\neg\alpha}$.*

Proof. By Lemma 13, operations are well defined. The fact that $\langle \mathbb{W}, \wedge, \vee \rangle$is a distributive lattice follows from the rules of conjunction and disjunction in the usual way. The property $\overline{\alpha} \wedge \neg(\overline{\alpha} \wedge \overline{\beta}) \leq \neg\overline{\beta}$ follows from (iii), Lemma 3. We deduce the inequality $\neg(\overline{\alpha} \vee \overline{\beta}) \leq \neg\overline{\alpha} \wedge \neg\overline{\beta}$ from (v), Lemma 3 and (CI). The other inequality: $\neg\overline{\alpha} \wedge \neg\overline{\beta} \leq \neg(\overline{\alpha} \vee \overline{\beta})$ is proved as follows: by (ii) and (vi) respectively we have: $(\neg\alpha \wedge \neg\beta) \wedge (\alpha \vee \beta) \vdash \neg(\gamma \wedge \neg\gamma)$ and $(\neg\alpha \wedge \neg\beta) \wedge (\alpha \vee \beta) \vdash \neg\neg(\gamma \wedge \neg\gamma)$. Then, by (NSI): $\neg\alpha \wedge \neg\beta \vdash \neg(\alpha \vee \beta)$. □

Lemma 15. *Let γ, $\varphi \in \mathcal{F}_{\mathcal{S}}$. Then, $\overline{\varphi}$ belongs to the principal filter generated by $\overline{\gamma}$ if and only if $\gamma \vdash \varphi$.*

8. Completeness

We define valuations on weak algebras in order to prove soundness and adequacy of the logic $\mathcal{S}$.

Definition 2. Let A be a weak algebra. An A-valuation is a function $v : Var \longrightarrow A$. A valuation v can be extended recursively to all formulas of $\mathcal{F}_{\mathcal{S}}$ in the usual way.

We say that an A-valuation v_A satisfies φ if $v_A(\varphi) = 1$. The formula φ is A–valid, noted $\models_A \varphi$, if any A-valuation satisfies φ. If $\models_A \varphi$ for every A then we note $\models \varphi$. A formula φ is an A*–semantic consequence* of a set of formulas $\Gamma = \{\gamma_1, \gamma_2, ..., \gamma_n\}$ if for any valuation v on A,$v(\gamma_1 \wedge \gamma_2 \wedge ... \wedge \gamma_n) \leq v(\varphi)$. We denote this fact by $\Gamma \models_A \varphi$ and $\Gamma \models \varphi$ if the condition holds for every A. In particular, if γ_1, γ_2, $...\gamma_n$ are A-valid formulas (respectively valid formulas), then φ is A-valid (respectively valid).

Theorem 16. *Let* $\Gamma \cup \{\varphi\} \subseteq \mathcal{F}_{\mathcal{S}}$. *If* $\Gamma \vdash \varphi$, *then* $\Gamma \models \varphi$.

Proof. It follows straightforwardly by induction, so we only consider the case of the rule (NSI): let v be a valuation such that $v(\gamma \wedge \alpha) \leq v(\beta \wedge \neg\beta)$. Then, by Lemma 5 (i), we have that $v(\gamma) \leq \neg v(\alpha)$. □

Theorem 17. *Let* $\Gamma \cup \{\varphi\} \subseteq \mathcal{F}_{\mathcal{S}}$. *If* $\Gamma \models \varphi$, *then* $\Gamma \vdash \varphi$.

Proof. By Lemma 2.2 we do not loose generality if we suppose $\Gamma = \{\gamma\}$.

Let γ and φ be formulas such that $\gamma \models \varphi$. Let p be the canonical map, $p : \mathcal{F}_{\mathcal{S}} \longrightarrow \mathbb{W}$. Take $\mathbb{F} = [\overline{\gamma})$, that is, $\mathbb{F}$ is the principal filter generated by $\overline{\gamma}$ in $\mathbb{W}$. Let $R_{\mathbb{F}}$ be the congruence associated with $\mathbb{F}$ and let q be the canonical map $q : \mathbb{W} \longrightarrow \mathbb{W}/R_{\mathbb{F}}$. Suppose $\gamma \nvdash \varphi$. Then, by Lemma 15, $\overline{\varphi} \notin \mathbb{F}$. But $q \circ p$ is a valuation, $(q \circ p)(\gamma) = q(\overline{\gamma}) = 1$ and $(q \circ p)(\varphi) = q(\overline{\varphi}) \neq 1$, absurd. □

9. Logical Glivenko theorem

In this section we prove Glivenko theorem for the logic $\mathcal{S}$ from the algebraic Glivenko theorem given in section 6. By Lemma 2.2 we do not loose generality if we suppose $\Gamma = \{\alpha\}$. In which follows $\mathcal{C}$ is classical propositional logic in the same language as $\mathcal{S}$ and deduction is given by Gentzen rules (see e.g. [8], system **LK**).

Theorem 18. *If* $\alpha \vdash_{\mathcal{C}} \beta$*, then* $\neg\neg\alpha \vdash_{\mathcal{S}} \neg\neg\beta$.

Proof. If $\alpha \vdash_{\mathcal{C}} \beta$, then for every Boolean algebra and valuation v, $v(\neg\alpha \vee \beta) = \neg v(\alpha) \vee v(\beta) = 1$. Let A be a weak algebra, $Reg(A)$ the Boolean algebra of regular elements (see section 6), $v(\alpha) = x$ and $v(\beta) = y$. Then $\neg x \vee y = 1$ in $Reg(A)$, which implies $\neg\neg(\neg x \vee y) = 1$ in A, by the algebraic Glivenko Theorem. Using Lemma 5 (i) it follows that $v(\neg\neg\alpha) \leq v(\neg\neg\beta)$. By completeness, $\neg\neg\alpha \vdash_{\mathcal{S}} \neg\neg\beta$. □

Let $\mathcal{C}$' (respectively $\mathcal{S}$') be the fragment of $\mathcal{C}$ (respectively $\mathcal{S}$) obtained by dropping disjunction from the language and the corresponding rules. Then, $\mathcal{S}$' is also obtained by dropping disjunction and conditional from $\mathcal{M}$. It follows (Luiz Carlos Pereira personal communication):

Corollary 19. *Let* $\alpha \in \mathcal{F}_{\mathcal{S}'}$. *If* $\vdash_{\mathcal{C}'} \alpha$*, then* $\vdash_{\mathcal{S}'} \alpha$.

Proof. Gödel noted that α must be of the form $\neg\beta_1 \wedge \neg\beta_2 \wedge ... \neg\beta_n$ (see [4]). To see this use induction on the complexity of α: the only non trivial case is the conjunction, but for $\alpha = \gamma \wedge \delta$, if $\vdash_{\mathcal{C}'} \gamma \wedge \delta$ then $\vdash_{\mathcal{C}'} \gamma$ and $\vdash_{\mathcal{C}'} \delta$; apply induction hypothesis. Now, if $\vdash_{\mathcal{C}'} \neg\beta_1 \wedge \neg\beta_2 \wedge ... \neg\beta_n$, then $\vdash_{\mathcal{C}'} \neg\beta_i$ for every $i = 1, ..., n$ and, using the above logical Glivenko theorem for $\mathcal{S}$, we have $\vdash_{\mathcal{S}'} \neg\beta_i$. So $\vdash_{\mathcal{S}'} \alpha$. □

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